J. Differential Equations 253 (2012) 2025–2063

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Existence and nonexistence of positive steady states in multi-species phytoplankton dynamics ☆

Linfeng Mei [a,*], Xiaoyan Zhang [b]

[a] Department of Mathematics, Henan Normal University, Xinxiang, Henan 453007, PR China
[b] School of Mathematics, Shandong University, Jinan, Shandong 250100, PR China

ARTICLE INFO

Article history:
Received 7 October 2011
Revised 3 June 2012
Available online 27 June 2012

MSC:
35J55
35J65
92D25

Keywords:
Existence
Nonexistence
Multi-species
Phytoplankton
Nonlocal reaction–diffusion equation

ABSTRACT

In this paper we study a nonlocal reaction–diffusion–advection system modeling the growth of multiple competitive phytoplankton species in a vertical water column with incomplete mixing. We find that when the diffusion of the system is large, there is no positive steady states, and when the diffusion is not large, there exists at least one positive steady states under certain conditions. The main tools we use are the fixed point index theory, a refined comparison theorem and fine properties of the principal eigenvalues.

1. Introduction

In this paper, we study an n-species reaction–diffusion–advection system proposed by Huisman et al. in [15] modeling the growth of competitive phytoplankton species in a vertical water column

$$(u_i)_t = D_i(u_i)_{xx} - \alpha_i(u_i)_x + \left[g_i\big(I(x,t,u)\big) - d_i\right]u_i, \quad 0 < x < L,\ t > 0,\ i = 1, 2, \ldots, n, \tag{1.1}$$

with no-flux boundary conditions

☆ L. Mei was supported by National Natural Science Foundation of China (10971046). X. Zhang was supported by the National Natural Science Foundation of China (10701049) and Independent Innovation Foundation of Shandong University (2012TS020) and Shandong Provincial Natural Science Foundation (ZR2012AQ007).

* Corresponding author.
E-mail addresses: sdaimei@hotmail.com (L. Mei), zxysd@sdu.edu.cn (X. Zhang).

http://dx.doi.org/10.1016/j.jde.2012.06.011

$$D_i(u_i)_x(0,t) - \alpha_i u_i(0,t) = D_i(u_i)_x(L,t) - \alpha_i u_i(L,t) = 0, \quad t \geqslant 0,\ i = 1, 2, \ldots, n, \tag{1.2}$$

and initial conditions

$$u_i(x,0) = u_i^0(x) \geqslant\not\equiv 0, \quad 0 \leqslant x \leqslant L,\ i = 1, 2, \ldots, n. \tag{1.3}$$

Here u_i is the population density of the phytoplankton species i, $D_i > 0$ is the diffusion coefficient caused by the water turbulence, $\alpha_i \in R^1$ is the sinking ($\alpha_i > 0$) or buoyant ($\alpha_i < 0$) velocity, $d_i > 0$ is the loss rate, $L > 0$ is the depth of the water column. The light distribution function $I(x,t,u)$ takes the form

$$I(x,t,u) = I_0 e^{-k_0 x} \exp\left(-\sum_{j=1}^{n} k_j \int_0^x u_j(s,t)\,ds\right),$$

where $I_0 > 0$ is the incident light intensity, $k_0 > 0$ is the background turbidity, k_i is the absorption coefficient of the phytoplankton species i. The term $g_i(I)$ represents the specific growth rate of the phytoplankton species i, which is a function of the light intensity $I(x,t,u)$. In this model ample nutrient supply is assumed so that the phytoplankton growth is only limited by the light intensity. g_i $(i = 1, 2, \ldots, n)$ are smooth functions satisfying

$$\begin{cases} g_i(0) = 0, \quad g_i'(I) > 0 \quad \text{for } I > 0, \\ \text{there exist constants } \sigma_i > 0 \text{ such that } g_i(I) \leqslant \sigma_i I \text{ for small } I > 0. \end{cases} \tag{1.4}$$

Under assumption (1.4), we have

$$\int_0^\infty g_i\left(e^{-\sigma x}\right) dx < \infty \quad \text{for any constant } \sigma > 0.$$

Typical examples of g_i include

$$g_i(I) = \frac{m_i I}{\delta_i + I},$$

and

$$g_i(I) = \frac{m_i}{\delta_i}\left(1 - e^{-\delta_i I}\right),$$

where m_i, δ_i, $i = 1, 2, \ldots, n$, are positive constants.

Phytoplankton refer to microscopic plant-like photosynthetic organisms that drift in lakes and oceans. They grow abundantly around the globe and are the foundation of the marine food chain. Phytoplankton depend on nutrients and light for their growth. In some water systems, there are ample nutrients supply and the phytoplankton species solely compete for light. These are called eutrophic environments. In the water column, phytoplankton are not only diffused by water turbulence, but also sink or buoy, according to whether they are heavier than water or not.

Most of the mathematical treatments of (1.1)–(1.3) focused on one and two species. For one species self-shading model (i.e. $k_0 = 0$) it was studied by Shigesada and Okubo in [23] as early as in 1981. The existence, uniqueness and the global stability of the steady state for one species model of infinite deep water ($L = \infty$) have been established in [19,23]. For the one species $k_0 > 0$ case, it is indicated in [10] that the conditions for phytoplankton bloom development can be characterized by critical water depth and critical values of the vertical turbulent diffusion coefficient. When there is no sinking or buoyancy, a complete description of the long-time dynamical behavior of one species model, the existence of positive steady states and uniform persistence of the dynamical system for the two species were studied in [8]. In [14], the existence and uniqueness of positive steady states of (1.1)–(1.3) (when $n = 1$) was established and the critical death rate, critical water column depth, critical sinking or buoyant coefficient and critical turbulent diffusion rate were studied respectively. In a recent paper [9], the global dynamics of the model (1.1)–(1.3) with $n = 1$ was investigated for the general case $D = D(x) \in C^1([0, L])$, $\alpha = \alpha(x) \in C^1([0, L])$ and the asymptotic profiles of the positive steady state solution for small diffusion, large diffusion and deep water column were examined respectively under the case $D(x) \equiv D > 0$, $\alpha(x) \equiv \alpha > 0$. Some rigorous mathematical treatment of systems with nutrients and light can be found in [6,7] and [22].

While much mathematical theory has been established for one or two species phytoplankton systems, little is achieved in this direction for three or more species ones. In multi-species micro-organism competitions, it was predicted in Huisman and Weissing [16,17,24] that interspecific competition in well-mixed environment leads to competitive exclusion, similar to that in [12,13]. However it is widely observed in multi-species phytoplankton communities that these communities often appear to violate the competitive exclusion principle [1]. This phenomenon is the so-called paradox of plankton, see [18] and references therein. Some results on the coexistence and persistence of two competing phytoplankton species have already been established mathematically in [8]. In this paper we extend the results in [8] to reveal a general phenomenon. For three species, we show that when the turbulence diffusion rates D_i, $i = 1, 2, 3$, are very large, generically there are no possibility of coexistence of multiple phytoplankton species in the same water column, see Theorem 3.1. However when the turbulence diffusion is not large, multiple phytoplankton species can coexist in the same water column, in certain parameter ranges, see Theorems 3.4 and 3.5. Moreover, we show that this phenomenon remains valid for any $n \geqslant 3$ systems of competing phytoplankton species of the form (1.1)–(1.3).

For reaction–diffusion systems of multi-species Lotka–Volterra type, there are already many results. Dancer and Du in [4] first developed a theory on fixed point index calculation, and then applied it to study the existence of positive solutions to various three species reaction–diffusion systems. See also results in Du [5] concerning the existence of positive periodic solutions for a competitor–competitor–mutualist model with diffusion. Other results concerning three species systems can be found in [11, 20,21] and the references therein. For our system, the mathematical treatment is much more difficult. The difficulties are twofold. One is the nonlocal terms involved, which, among other difficulties, makes the usual comparison principle difficult to use. The other is the fact there are fewer parameters in the system than in the classical Lotka–Volterra type competition models, and hence sufficient conditions for coexistence are more difficult to formulate. Because of these difficulties, we need much more involved analysis on the solutions of the system in order to make the abstract tools applicable.

Before continuing our discussion, we do some rescaling to simplify our system (1.1)–(1.3). Replacing x with Lx, $u_i(\cdot)$ with $k_i^{-1}u_i(L\cdot)$, D_i with L^2D_i, α_i with $L\alpha_i$, k_i with k_iL and $g_i(I_0\cdot)$ with $g_i(\cdot)$, we may assume that u_i satisfies the modified system

$$(u_i)_t = D_i(u_i)_{xx} - \alpha_i(u_i)_x + \big[g_i\big(I(x,t,u)\big) - d_i\big]u_i, \quad 0 < x < 1,\ t > 0,\ i = 1, 2, \ldots, n, \tag{1.5}$$

with no-flux boundary conditions

$$D_i(u_i)_x(0,t) - \alpha_i u_i(0,t) = D_i(u_i)_x(1,t) - \alpha_i u_i(1,t) = 0, \quad t \geqslant 0,\ i = 1, 2, \ldots, n, \tag{1.6}$$

and initial conditions

$$u_i(x,0) = u_i^0(x) \geqslant\not\equiv 0, \quad 0 \leqslant x \leqslant 1,\ i = 1, 2, \ldots, n, \tag{1.7}$$

with

$$I(x,t) = I(x,t,u) = e^{-k_0 x} \exp\left(- \int_0^x \left[\sum_{j=1}^n u_j(s,t) \right] ds \right).$$

Through the change of variables

$$u_i(x,t) = v_i(x,t) e^{(\alpha_i/D_i)x},$$

we arrive at an equivalent system

$$(v_i)_t = D_i (v_i)_{xx} + \alpha_i (v_i)_x + \left[g_i\big(I(x,t)\big) - d_i \right] v_i, \quad 0 < x < 1,\ t > 0,\ i = 1, 2, \ldots, n, \tag{1.8}$$

with homogeneous Neumann boundary conditions

$$(v_i)_x(0,t) = (v_i)_x(1,t) = 0, \quad t \geqslant 0,\ i = 1, 2, \ldots, n, \tag{1.9}$$

and initial conditions

$$v_i(x,0) = v_i^0(x) \geqslant\not\equiv 0, \quad 0 \leqslant x \leqslant 1,\ i = 1, 2, \ldots, n, \tag{1.10}$$

where

$$I(x,t) = e^{-k_0 x} \exp\left(- \int_0^x \left[\sum_{j=1}^n e^{(\alpha_j/D_j)s} v_j(s,t) \right] ds \right).$$

We will use the $n = 3$ case to illustrate our method. The extension of the method to the general $n \geqslant 3$ case is straightforward. To make our results more understandable, we also include here the two species case (with drifting term as opposed to the no drifting case treated in [8]).

The main structure of the paper is as following. In Section 2, we generalize the existence results of [8] (two species system without drifting) to our setting. Section 3 is devoted to the study of the existence and nonexistence of positive steady state of our system for $n = 3$. In Section 4 we use fixed point index calculation to prove a group of explicit conditions for the existence of positive solutions, on which Section 3 is based. In the final section, Section 5, we extend the results before to the general $n(\geqslant 3)$ species case.

2. Steady states of the 2-species model

In this section we study the steady states of the two species model. Here we assume the diffusion and sinking (buoyant) rates of the phytoplankton species depend on the water depth. This makes little difference as far as the existence of the positive steady states is concerned, but the generalization may find use elsewhere. Namely we consider the system

$$\begin{cases} (u_i)_t = \big(D_i(x)(u_i)_x - \alpha_i(x) u_i\big)_x + \big(g_i\big(I(x,t)\big) - d_i\big) u_i, & 0 < x < 1,\ t > 0 \\ D(0)(u_i)_x(0,t) - \alpha_i(0) u_i(0,t) = D_i(0)(u_i)_x(1,t) - \alpha_i(h) u_i(1,t) = 0, & t \geqslant 0, \\ u_i(x,0) = u_i^0(x) \gneqq 0, & 0 \leqslant x \leqslant 1,\ i = 1, 2, \end{cases} \tag{2.1}$$

where $g_i \in C^1([0,\infty))$ satisfies

$$g_i(0) = 0 \quad \text{and } g_i \text{ is strictly increasing,} \tag{2.2}$$

$$I(x,t) = e^{-k_0 x} \exp\left(-\int_0^x \left[u_1(s,t) + u_2(s,t)\right] ds\right), \tag{2.3}$$

$D_i(x), \alpha_i(x) \in C^1([0,1])$, $D_i(x) > 0$, k_0 is positive constant and $d_1, d_2 \in (0,\infty)$ are parameters.

For $i = 1, 2$, let

$$u_i(x,t) = e^{R_i(x)} v_i(x,t)$$

with

$$R_i(x) = \int_0^x \frac{\alpha_i(s)}{D_i(s)}\, ds.$$

Then (2.1) becomes

$$\begin{cases} (v_i)_t = e^{-R_i(x)}\big(D_i(x)e^{R_i(x)}(v_i)_x\big)_x + \big(g_i\big(I(x,t)\big) - d_i\big)v_i, & 0 < x < 1,\ t > 0, \\ (v_i)_x(0,t) = (v_i)_x(1,t) = 0, & t \geqslant 0, \\ v_i(x,0) = e^{-R_i(x)}u_i^0(x) =: v_i^0(x) \gneqq 0, & 0 \leqslant x \leqslant 1, \end{cases} \tag{2.4}$$

where

$$I(x,t) = e^{-k_0 x} \exp\left(-\int_0^x \left[e^{R_1(s)} v_1(s,t) + e^{R_2(s)} v_2(s,t)\right] ds\right).$$

The corresponding steady state system is

$$\begin{cases} -e^{-R_1(x)}\big(D_1(x)e^{R_1(x)}v_1'\big)' = \big(g_1\big(I(x)\big) - d_1\big)v_1, & 0 < x < 1, \\ -e^{-R_2(x)}\big(D_2(x)e^{R_2(x)}v_2'\big)' = \big(g_2\big(I(x)\big) - d_2\big)v_2, & 0 < x < 1, \\ v_i'(0) = v_i'(1) = 0, & i = 1, 2, \end{cases} \tag{2.5}$$

where

$$I(x) = e^{-k_0 x} \exp\left(-\int_0^x \left[e^{R_1(s)} v_1(s) + e^{R_2(s)} v_2(s)\right] ds\right).$$

We want to find sufficient conditions for (2.5) to have at least one positive solution.

For a function $\Psi \in C([0,1])$, let $\lambda_1^{(i)}(\Psi)$, $i = 1, 2$, be the first eigenvalue of the following eigenvalue problem

$$-e^{-R_i(x)}\big(D_i(x)e^{R_i(x)}\varphi'\big)' + \Psi(x)\varphi = \lambda\varphi, \quad 0 < x < 1,\ \varphi'(0) = \varphi'(1) = 0. \tag{2.6}$$

Clearly $\lambda_1^{(i)}(0) = 0$, $i = 1, 2$.

Define

$$d_i^* = -\lambda_1^{(i)}\left(-g_i\left(e^{-k_0 x}\right)\right).$$

Nonnegative solutions of (2.5) can be classified into three classes: The unique trivial solution $(v_1, v_2) = (0, 0)$, which exists for all $d_1, d_2 \in R$. Two semitrivial solutions $(v_1, v_2) = (0, v_{d_2}^*)$ and $(v_1, v_2) = (v_{d_1}^*, 0)$, the former exists for $d_2 \in (0, d_2^*)$ and the latter exists for $d_1 \in (0, d_1^*)$, where $v_{d_1}^*, v_{d_2}^*$ denote the unique positive steady state for the v_1 and v_2 equations respectively, guaranteed by Theorem 2.1 of [9]. The third class are positive solutions (v_1, v_2) with $v_1 > 0$ and $v_2 > 0$ in $[0, 1]$, which are the main interest here.

A necessary condition for the existence of a positive solution to (2.5) can be easily observed. Suppose that (v_1, v_2) is a positive solution of (2.5). Then from the equation for v_1 we obtain

$$-d_1 = \lambda_1^{(1)}\left(-g_1\left(e^{-k_0 x - \int_0^x [e^{R_1(s)} v_1(s) + e^{R_2(s)} v_2(s)]\, ds}\right)\right) \in \left(-d_1^*, 0\right).$$

That is $d_1 \in (0, d_1^*)$. Similarly from the equation for v_2 we deduce $d_2 \in (0, d_2^*)$. Thus for (2.5) to have a positive solution we necessarily have

$$0 < d_1 < d_1^*, \qquad 0 < d_2 < d_2^*. \tag{2.7}$$

On the other hand, we have the following

Theorem 2.1. *Let $v_{d_i}^*$, $d_i \in (0, d_i^*)$, $i = 1, 2$ be the unique positive solution of the problem*

$$\begin{cases} -e^{-R_i(x)}\left(D_i(x) e^{R_i(x)} v'\right)' = \left[g_i\left(e^{-k_0 x - \int_0^x e^{R_i(s)} v(s)\, ds}\right) - d_i\right] v, & 0 < x < 1, \\ v'(0) = v'(1) = 0. \end{cases} \tag{2.8}$$

If

$$\begin{cases} 0 < d_1 < -\lambda_1^{(1)}\left[-g_1\left(e^{-k_0 x - \int_0^x e^{R_2(s)} v_{d_2}^*(s)\, ds}\right)\right] =: \tilde{d}_1, \\ 0 < d_2 < -\lambda_1^{(2)}\left[-g_2\left(e^{-k_0 x - \int_0^x e^{R_1(s)} v_{d_1}^*(s)\, ds}\right)\right] =: \tilde{d}_2, \end{cases} \tag{2.9}$$

then (2.5) has at least one positive solution.

To prove Theorem 2.1, let $E = C([0, 1])$ and let P be the usual positive cone in E: $P = \{v \in E\colon v(x) \geqslant 0 \text{ in } [0, 1]\}$. We define

$$A(v_1, v_2) = \left(A_1(v_1, v_2), A_2(v_1, v_2)\right),$$

where

$$\begin{aligned} &A_1(v_1, v_2) = L_1 \circ G_1(d_1, v_1, v_2), \qquad A_2(v_1, v_2) = L_2 \circ G_2(d_2, v_1, v_2), \\ &G_1(d_1, v_1, v_2)(x) = \left[d_1^* - d_1 + g_1\left(e^{-k_0 x - \int_0^x (e^{R_1} v_1 + e^{R_2} v_2)\, ds}\right)\right] e^{R_1(x)} v_1(x), \\ &G_2(d_2, v_1, v_2)(x) = \left[d_2^* - d_2 + g_2\left(e^{-k_0 x - \int_0^x (e^{R_1} v_1 + e^{R_2} v_2)\, ds}\right)\right] e^{R_2(x)} v_2(x), \end{aligned}$$

and for $i = 1, 2$, L_i is the solution operator for the problem

$$-\left(D_i(x)e^{R_i(x)}v'\right)' + d_i^* e^{R_i(x)} v = f_i(x), \qquad v_i'(0) = v_i'(1) = 0,$$

namely $v = L_i(f_i)$. It is easily seen that (v_1, v_2) solves (2.5) if and only if $(v_1, v_2) = A(v_1, v_2)$.

By standard elliptic regularity theory we know that $A : E \times E \to E \times E$ is completely continuous. Moreover, by the strong maximum principle and the fact that

$$d_i^* - d_i + g_i\left(e^{-k_0 x - \int_0^x (e^{R_1} v_1 + e^{R_2} v_2)\,ds}\right) > 0 \quad \text{in } [0,1],$$

we find that $v_i \in \dot{P} := P \setminus \{0\}$ implies $A_i(v_1, v_2) \in P^\circ := \{v \in P\colon\ v(x) > 0 \text{ in } [0,1]\}$. Thus we have

$$A(P \times P) \subset P \times P, \qquad A(\dot{P} \times \dot{P}) \subset P^\circ \times P^\circ,$$
$$A(\dot{P} \times P) \subset P^\circ \times P, \qquad A(P \times \dot{P}) \subset P \times P^\circ.$$

To use the topological degree (fixed point index) argument, we need some preparation.

Lemma 2.2. *Let $\bar{d}_1 \in (0, d_1^*)$, $\bar{d}_2 \in (0, d_2^*)$ be two constants. Then there exist positive constants $C_1 = C_1(\bar{d}_1, \bar{d}_2)$, $C_2 = C_2(\bar{d}_1, \bar{d}_2)$ such that for any nonnegative solution (v_1, v_2) of (2.5) corresponding to (d_1, d_2) with $d_1 \geqslant \bar{d}_1$, $d_2 \geqslant \bar{d}_2$, one has the estimate*

$$\|v_i\|_\infty \leqslant C_i, \quad i = 1, 2. \tag{2.10}$$

Proof. Argue indirectly. Suppose there is a sequence of (d_1, d_2), say (d_{1n}, d_{2n}) and the corresponding nonnegative solutions v_{1n}, v_{2n} of (2.5) such that $\|v_{1n}\|_\infty + \|v_{2n}\|_\infty \to \infty$. Without loss of generality, we assume $\|v_{1n}\|_\infty \to \infty$. Set $\tilde{v}_{1n} = v_{1n}/\|v_{1n}\|_\infty$. Then $\tilde{v}_{1n}$ satisfies

$$-e^{R_1(x)}\left(D_1(x)e^{R_1(x)}\tilde{v}_{1n}'\right)' = \left[g_1\left(I_n(x)\right) - d_{1n}\right]\tilde{v}_{1n}, \qquad \tilde{v}_{1n}'(0) = \tilde{v}_{1n}'(1) = 0, \tag{2.11}$$

where

$$I_n(x) = e^{-k_0 x} \exp\left(-\|v_{1n}\|_\infty \int_0^x e^{R_1(s)}\tilde{v}_{1n}(s)\,ds - \int_0^x e^{R_2(s)} v_{2n}(s)\,ds\right).$$

The right-hand side of (2.11) is clearly uniformly bounded. By the standard elliptic regularity, we may assume, by passing to a subsequence, $\tilde{v}_{1n} \to v_0$ in $C^1([0,1])$. We may also assume $g_1(I_n) \to g_0$ weakly in $L^2((0,1))$, $d_{1n} \to d_0 \geqslant \bar{d}_1$. Moreover v_0 satisfies (in the weak sense)

$$-e^{R_1(x)}\left(D_1(x)e^{R_1(x)}v_0'\right)' = \left[g_0(x) - d_0\right]v_0, \qquad v_0'(0) = v_0'(1) = 0, \qquad \|v_0\|_\infty = 1.$$

By the maximum principle $\min_{x\in[0,1]} v_0(x) > c_0 > 0$ for some constant c_0. Therefore $\lim_{n\to\infty} v_{1n}(x) = \infty$ uniformly, and $g_0(x) = \lim_{n\to\infty} g_1(I_n(x)) = 0$. It follows $-d_0 = -\lambda_1(0) = 0$, contradicting $d_0 \geqslant \bar{d}_1$. The contradiction proves the lemma. □

The Frechet derivative of $A(v_1, v_2)$ with respect to (v_1, v_2) at $(v_{d_1}^*, 0)$ and at $(0, v_{d_2}^*)$, and the associated eigenvalue problems play a crucial role. We will denote these derivatives by $A'_{(v_1,v_2)}(v_{d_1}^*, 0)$ and $A'_{(v_1,v_2)}(0, v_{d_2}^*)$, respectively, and the associated eigenvalue problems are

$$A'_{(v_1,v_2)}\left(v_{d_1}^*, 0\right)(m_1, m_2) = \xi(m_1, m_2), \tag{2.12}$$

and

$$A'_{(v_1,v_2)}\big(0, v^*_{d_2}\big)(m_1, m_2) = \eta(m_1, m_2). \tag{2.13}$$

A direct calculation show that $\eta = 1$ is an eigenvalue of (2.13) if and only if the following problem has a solution $(m_1, m_2) \neq (0, 0)$:

$$\begin{cases} -e^{-R_1(x)}\big(D_1(x)e^{R_1(x)}m_1'\big)' = \big[g_1\big(\sigma_2(x)\big) - d_1\big]m_1, & x \in (0,1), \\ -e^{-R_2(x)}\big(D_2(x)e^{R_2(x)}m_2'\big)' = \big[g_2\big(\sigma_2(x)\big) - d_2\big]m_2 - \delta_2(x), & x \in (0,1), \\ m_1' = m_2' = 0, & x = 0, 1, \end{cases} \tag{2.14}$$

where

$$\delta_2(x) = g_2'\big(\sigma_2(x)\big)\sigma_2(x)v^*_{d_2}(x)\int_0^x \big[e^{R_1(s)}m_1(s) + e^{R_2(s)}m_2(s)\big]\,ds, \quad \sigma_2(x) = e^{-k_0x - \int_0^x e^{R_2(s)}v^*_{d_2}(s)\,ds}.$$

Similarly, if we define

$$\sigma_1(x) = e^{-k_0x - \int_0^x e^{R_1(s)}v^*_{d_1}(s)\,ds},$$

then $\xi = 1$ is an eigenvalue of (2.13) if and only if the following problem has a solution $(m_1, m_2) \neq (0, 0)$:

$$\begin{cases} -e^{-R_1(x)}\big(D_1(x)e^{R_1(x)}m_1'\big)' = \big[g_1\big(\sigma_1(x)\big) - d_1\big]m_1 - \delta_1(x), & x \in (0,1), \\ -e^{-R_2(x)}\big(D_2(x)e^{R_2(x)}m_2'\big)' = \big[g_2\big(\sigma_1(x)\big) - d_2\big]m_2, & x \in (0,1), \\ m_1' = m_2' = 0, & x = 0, 1, \end{cases} \tag{2.15}$$

where

$$\delta_1(x) = g_1'\big(\sigma_1(x)\big)\sigma_1(x)v^*_{d_1}(x)\int_0^x \big[e^{R_1(s)}m_1(s) + e^{R_2(s)}m_2(s)\big]\,ds.$$

The following lemma holds the key for solving (2.14) and (2.15).

Lemma 2.3. *Let $i \in \{1, 2\}$. If $\psi \in C^2([0, 1])$ satisfies*

$$\begin{cases} -e^{-R_i(x)}\big(D_i(x)e^{R_i(x)}\psi'\big)' = \big[g_i\big(\sigma_i(x)\big) - d_i\big]\psi \\ -g_i'\big(\sigma_i(x)\big)\sigma_i(x)v^*_{d_i}(x)\displaystyle\int_0^x e^{R_i(s)}\psi(s)\,ds, & x \in (0,1), \\ \psi'(0) = \psi'(1) = 0, \end{cases} \tag{2.16}$$

then $\psi \equiv 0$.

Proof. We argue indirectly. Suppose $\psi \not\equiv 0$ solves (2.16). We first claim that $\psi(0) \neq 0$. Otherwise, define

$$\xi(x) = \int_0^x e^{R_i(s)} \psi(s)\, ds, \qquad \eta(x) = D_i(x) e^{R_i(x)} \psi'(x).$$

Then $(\xi(x), \psi(x), \eta(x))$ is a solution of the ODE system

$$\begin{cases} \xi' = e^{R_i(x)} \psi, \\ \psi' = D_i^{-1}(x) e^{-R_i(x)} \eta, \\ \eta' = -\left[g_i\left(\sigma_i(x)\right) - d_i\right] e^{R_i(x)} \psi + g_i'\left(\sigma_i(x)\right) \sigma_i(x) v_{d_i}^*(x) e^{R_i(x)} \xi, \end{cases} \tag{2.17}$$

with the initial condition $(\xi(0), \psi(0), \eta(0)) = (0, 0, 0)$. Clearly $(\xi, \psi, \eta) \equiv (0, 0, 0)$ is the unique solution of this initial value ODE problem. Hence $\psi \equiv 0$, contradicting our assumption that $\psi \not\equiv 0$.

Without loss of generality we may assume that $\psi(0) > 0$. Next we claim that $\psi(x)$ changes sign in $(0, 1)$. Otherwise $\psi(x) \geqslant, \not\equiv 0$ in $[0, 1]$. Multiplying the first equation in (2.16) by $e^{R_i(x)} v_{d_i}^*$ and integrating it over $[0, 1]$, we deduce

$$\begin{aligned} \int_0^1 D_i(x) e^{R_i(x)} \psi'(x) \left(v_{d_i}^*\right)'(x)\, dx = {} & \int_0^1 \left[g_i\left(\sigma_i(x)\right) - d_i\right] e^{R_i(x)} \psi(x) v_{d_i}^*(x)\, dx \\ & - \int_0^1 g_i'\left(\sigma_i(x)\right) \sigma_i(x) e^{R_i(x)} \left[v_{d_i}^*(x)\right]^2 \int_0^x e^{R_i(s)} \psi(s)\, ds\, dx. \end{aligned} \tag{2.18}$$

$v_{d_i}^*$ satisfies

$$\begin{cases} -e^{-R_i(x)} \left(D_i(x) e^{R_i(x)} \left(v_{d_i}^*\right)'\right)' = \left(g_1\left(\sigma_i(x)\right) - d_i\right) v_{d_i}^*, & 0 < x < 1, \\ \left(v_{d_i}^*\right)'(0) = \left(v_{d_i}^*\right)'(1) = 0. \end{cases} \tag{2.19}$$

Multiplying the first equation of (2.19) by $e^{R_i(x)} \psi$ and integrating it over $[0, 1]$, we deduce

$$\int_0^1 D_i(x) e^{R_i(x)} \psi'(x) \left(v_{d_i}^*\right)'(x)\, dx = \int_0^1 \left[g_i\left(\sigma_i(x)\right) - d_i\right] e^{R_i(x)} \psi(x) v_{d_i}^*(x)\, dx. \tag{2.20}$$

From (2.18) and (2.20) we readily have

$$\int_0^1 g_i'\left(\sigma_i(x)\right) \sigma_i(x) e^{R_i(x)} \left[v_{d_i}^*(x)\right]^2 \int_0^x e^{R_i(s)} \psi(s)\, ds\, dx = 0.$$

But the integrand function in the last identity is clearly nonnegative and not identically zero in $[0, 1]$. Hence the integral should be positive. This contradiction shows that $\psi(x)$ changes sign in $(0, 1)$

Let $x_0 \in (0, 1)$ be the first zero of $\psi(x)$, namely $\psi(x) > 0$ in $[0, x_0)$ and $\psi(x_0) = 0$. We now consider the eigenvalue problem

$$-e^{R_i(x)}\big(D_i(x)e^{R_i(x)}\phi'\big)' = \big[g_i\big(\sigma_i(x)\big) - d_i\big]\phi + \lambda\phi, \quad \text{in } (0, x_0),\ \phi'(0) = \phi(x_0) = 0. \tag{2.21}$$

We claim that the first eigenvalue λ_1 of this problem is positive. Indeed, let ϕ_1 be a positive eigenfunction corresponding to λ_1. Multiplying the first equation in (2.21) (with $\lambda = \lambda_1$, $\phi = \phi_1$) by $e^{R_i(x)}v^*_{d_i}$ and integrating it over $[0, x_0]$ we obtain

$$\begin{aligned} &-D_i(x_0)e^{R_i(x_0)}\phi_1'(x_0)v^*_{d_i}(x_0) + \int_0^{x_0} D_i(x)e^{R_i(x)}\phi_1'(x)\big(v^*_{d_i}\big)'(x)\,dx \\ &\quad = \int_0^{x_0} \big[g_i\big(\sigma_i(x)\big) - d_i\big]e^{R_i(x)}\phi_1(x)v^*_{d_i}(x)\,dx + \lambda_1\int_0^{x_0} e^{R_i(x)}\phi_1(x)v^*_{d_i}(x)\,dx. \end{aligned}$$

On the other hand, multiplying (2.19) by $e^{R_i(x)}\phi_1$ and integrating it over $[0, x_0]$, we obtain

$$\int_0^{x_0} D_i(x)e^{R_i(x)}\phi_1'(x)\big(v^*_{d_i}\big)'(x)\,dx = \int_0^{x_0} \big[g_i\big(\sigma_i(x)\big) - d_i\big]e^{R_i(x)}\phi_1(x)v^*_{d_i}(x)\,dx.$$

From the last two identities, we arrive at

$$\lambda_1\int_0^{x_0} e^{R_i(x)}\phi_1(x)v^*_{d_i}(x)\,dx = -D_i(x_0)e^{R_i(x_0)}\phi_1'(x_0)v^*_{d_i}(x_0).$$

Hence $\lambda_1 > 0$ is clear, since $v^*_{d_i}(x_0) > 0$, $\int_0^{x_0} e^{R_i(x)}\phi_1(x)v^*_{d_i}(x)\,dx > 0$ and $\phi_1'(x_0) < 0$ (by the Hopf lemma).

To obtain the desired contradiction, we now multiply the first equation in (2.21) (with $\lambda = \lambda_1$, $\phi = \phi_1$) by $e^{R_i(x)}\psi$ and then integrate it over $[0, x_0]$. Consequently we deduce

$$\int_0^{x_0} D_i(x)e^{R_i(x)}\phi_1'(x)\psi'(x)\,dx = \int_0^{x_0} \big[g_i\big(\sigma_i(x)\big) - d_i\big]e^{R_i(x)}\phi_1(x)\psi(x)\,dx + \lambda_1\int_0^{x_0} e^{R_i(x)}\phi_1(x)\psi(x)\,dx.$$

On the other hand, multiplying the first equation in (2.16) by $e^{R_i(x)}\phi_1$ and integrating it over $[0, x_0]$, we deduce

$$\begin{aligned} \int_0^{x_0} D_i(x)e^{R_i(x)}\phi_1'(x)\psi'(x)\,dx &= \int_0^{x_0} \big[g_i\big(\sigma_i(x)\big) - d_i\big]e^{R_i(x)}\phi_1(x)\psi(x)\,dx \\ &\quad - \int_0^{x_0} g_i'\big(\sigma_i(x)\big)\sigma_i(x)v^*_{d_i}(x)e^{R_i(x)}\phi_1(x)\int_0^{x} e^{R_i(s)}\psi(s)\,ds\,dx. \end{aligned}$$

From the last two equations we arrive at

$$\lambda_1 \int_0^{x_0} D_i(x)e^{R_i(x)}\phi_1(x)\psi(x)\,dx$$
$$= -\int_0^{x_0} g_i'\big(\sigma_i(x)\big)\sigma_i(x)v_{d_i}^*(x)e^{R_i(x)}\phi_1(x)\int_0^{x} e^{R_i(s)}\psi(s)\,ds\,dx. \tag{2.22}$$

Since $\lambda_1 > 0$ and $\psi(x) > 0$ in $[0, x_0)$, the left side of the above identity is positive. However, the integrand function in the right side of (2.22) is nonnegative and hence the right side of (2.22) is not positive. This contradiction completes the proof. □

Lemma 2.4. *Problem* (2.14) *has a solution* $(m_1, m_2) \neq (0, 0)$ *if and only if* $m_1 \neq 0$ *and*

$$-e^{-R_1(x)}\big(D_1(x)e^{R_1(x)}m_1'\big)' = \big[g_1\big(\sigma_2(x)\big) - d_1\big]m_1, \quad m_1'(0) = m_1'(1) = 0.$$

Moreover, with m_1 *given,* m_2 *can be uniquely solved from the second equation in* (2.14) *together with the Neumann boundary conditions.*

Similarly, (2.15) *has a solution* $(m_1, m_2) \neq (0, 0)$ *if and only if* $m_1 \neq 0$ *and*

$$-e^{-R_2(x)}\big(D_2(x)e^{R_2(x)}m_2'\big)' = \big[g_2\big(\sigma_1(x)\big) - d_2\big]m_2, \quad m_2'(0) = m_2'(1) = 0.$$

Moreover, with m_2 *given,* m_1 *can be uniquely solved from the first equation in* (2.15) *together with the Neumann boundary conditions.*

Proof. We only consider the statement for (2.14); the proof of that for (2.15) is analogous. Let (m_1, m_2) solves (2.14). If $m_1 = 0$, then by Lemma 2.3 we deduce $m_2 = 0$. Suppose now $m_1 \neq 0$. Then we can apply the Fredholm alternative for compact operators and Lemma 2.3 to conclude that the second equation in (2.14) together with the Neumann boundary conditions in uniquely solvable for any given m_1. □

We are now read to prove our main theorem in this section, namely Theorem 2.1.

Proof of Theorem 2.1. Define $\Omega = \Lambda \times U \times V$ with

$$U = \big\{v_1 \in P\colon \|v_1\|_\infty < C\big\}, \qquad V = \big\{v_2 \in P\colon \|v_2\|_\infty < C\big\},$$

where $C > 0$ is large enough such that (2.10) holds and $\|v_{d_1}^*\|_\infty < C$.

Let B_1 be a small ball in E containing $v_{d_1}^*$. Since $v_{d_1}^* \in P^\circ$, we may assume that $B_1 \subset P^\circ$. Then by Lemma 2.1 of [4] (i.e. Lemma 4.3 below), we have

$$\operatorname{index}_{P\times P}\big(A(d_2,\cdot),\big(v_{d_1}^*,0\big)\big) = \begin{cases} 0 & \text{if } r(L) > 1, \\ \deg_P\big(I - A_1(\cdot,0), B_1\big) & \text{if } r(L) < 1, \end{cases}$$

where $L = (A_2)'_{v_2}(v_{d_1}^*, 0)$ and $r(L)$ denotes the spectral radius of the linear operator L.

It is easily checked that $r(L) > 1$ if $d_2 < -\lambda_1^{(1)}(-g_2(\sigma_1(x))) = \tilde{d}_2$, and $r(L) < 1$ if $d_2 > -\lambda_1^{(1)}(-g_2(\sigma_1(x))) = \tilde{d}_2$. Thus

$$\operatorname{index}_{P\times P}\big(A(d_2,\cdot),\big(v_{d_1}^*,0\big)\big) = \begin{cases} 0 & \text{if } d_2 < \tilde{d}_2, \\ \deg_P\big(I - A_1(\cdot,0), B_1\big) & \text{if } d_2 > \tilde{d}_2. \end{cases}$$

We show next that

$$\deg_P\big(I - A_1(\cdot, 0), B_1\big) = 1.$$

Since $(u^*_{d_1}, 0)$ is the only fixed point of $A_1(\cdot, 0)$ in $B_1 \cap P^\circ$, we clearly have

$$\deg_P\big(I - A_1(\cdot, 0), B_1\big) = \mathrm{index}_P\big(A_1(\cdot, 0), v^*_{d_1}\big).$$

We will use a homotopy argument to $A_1(\lambda, v_1, 0) = L_1 \circ G_1(\lambda, v_1, 0)$ with $\lambda \in [d^*_1, d^*_1 + 1]$. By Theorem 2.1 of [9] we know that for $\lambda \in [d_1, d^*_1)$ the equation $A_1(\lambda, v, 0) = v$ has exactly two solutions in P: The trivial solution $v = 0$ and the unique positive solution $v = v_\lambda > 0$. For $\lambda \in [d^*_1, d^*_1 + 1]$, there is one solution in P: $v = 0$. Moreover, one easily sees that 0 is a linearized stable fixed point of $A_1(\lambda, \cdot, 0)$ when $\lambda > d^*_1$, and it is a linearized unstable fixed point when $\lambda < d^*_1$. It follows that

$$\mathrm{index}_P\big(A_1(\lambda, \cdot, 0), 0\big) = \begin{cases} 0 & \text{for } \lambda < d^*_1, \\ 1 & \text{for } \lambda > d^*_1. \end{cases}$$

Choose $C_0 > 0$ large enough such that $\|v_\lambda\|_\infty < C_0$ for $\lambda \in [d_1, d^*_1)$, and denote $P_{C_0} := \{v \in P\colon \|v\|_\infty < C_0\}$. Then by the homotopy invariance property of the topological degree, we find that $\deg_P(I - A_1(\lambda, \cdot, 0), P_{C_0})$ is well defined and its value does not depend on λ for $\lambda \in [d_1, d^*_1 + 1]$. By the additivity of the topological degree we have

$$\begin{aligned}\deg_P\big(I - A_1(\lambda, \cdot, 0), P_{C_0}\big) &= \mathrm{index}_P\big(A_1(\lambda, \cdot, 0), 0\big) + \mathrm{index}_P\big(A_1(\lambda, \cdot, 0), v_\lambda\big) \\ &= \mathrm{index}_P\big(A_1(\lambda, \cdot, 0), v_\lambda\big)\end{aligned}$$

for $\lambda \in [d_1, d^*_1)$, and

$$\deg_P\big(I - A_1(\lambda, \cdot, 0), P_{C_0}\big) = \mathrm{index}_P\big(A_1(\lambda, \cdot, 0), 0\big) = 1$$

for $\lambda \in (d^*_1, d^*_1 + 1]$. It follows that

$$\mathrm{index}_P\big(A_1(\lambda, \cdot, 0), v_\lambda\big) = 1$$

for $\lambda \in [d_1, d^*_1)$. Taking $\lambda = d_1$ we obtain

$$\deg_P\big(I - A_1(\cdot, 0), B_1\big) = \mathrm{index}_P\big(A_1(\lambda, \cdot, 0), v^*_{d_1}\big) = 1.$$

The proof of Theorem 2.1 is complete. $\square$

Before ending this section, we give some discussion on condition (2.9). The condition is rather implicit. We have showed that $d_1 \in (0, d^*_1)$ and $d_2 \in (0, d^*_2)$ is necessary for (2.5) to have a positive solution. Condition (2.9) is more restrictive than this necessary condition. Indeed we have

Proposition 2.5. *For fixed $d_1 \in (0, d^*_1)$, if $\delta > 0$ is small enough, then (2.5) has no positive solution if $d_2 \notin (\delta, d^*_2 - \delta)$.*

Proof. Otherwise, we can find $d_2^n \downarrow 0$ or $d_2^n \uparrow d_2^*$ and a positive solution (v_1^n, v_2^n) with $d_2 = d_2^n$. In the first case we define $\hat{v}_2^n = v_2^n / \|v_2^n\|_\infty$ and

$$f_n = \exp\left(-k_0 x - \int_0^x e^{R_1(s)} v_1^n(s)\,ds - \int_0^x e^{R_2(s)} v_2^n(s)\,ds\right),$$

and as before find that by passing to a subsequence $\hat{u}_i^n \to \hat{v}_i$ in $C^1([0,1])$ for $i = 1, 2$, $f_n \to f$ and $g_2(f_n) \to g_2(f)$ weakly in $L^2([0,1])$, and $\hat{v}_2$ is a positive solution to

$$-e^{-R_2(x)}\left(D_2(x)e^{R_2(x)}\hat{v}_2'\right)' = g_2(f)\hat{v}_2, \qquad \hat{v}'(0) = \hat{v}'(1) = 0. \tag{2.23}$$

Multiplying the first equation in (2.23) by $e^{R_2(x)}$ and integrating the resultant equation over $[0,1]$ we get

$$\int_0^1 g_2(f)(x)e^{R_2(x)}\hat{v}_2(x)\,dx = 0.$$

Since $\hat{u} > 0$ in $[0,h]$ and $g(f) \geqslant 0$ in $[0,1]$, the above identity implies $g_2(f) = 0$ a.e. in $[0,1]$. It follows that $f(x) = 0$ a.e. in $[0,1]$.

Now define $\hat{v}_1^n = v_1^n / \|v_1^n\|_\infty$ and we obtain from the equation for v_1^n that

$$-e^{-R_1(x)}\left(D_1(x)e^{R_1(x)}\left(\hat{v}_1^n\right)'\right)' = \left[g_1(f_n) - d_1\right]\hat{v}_1^n, \qquad \left(\hat{v}_1^n\right)'(0) = \left(\hat{v}_1^n\right)'(1) = 0.$$

As before by elliptic regularity and passing to a subsequence, $\hat{v}_1^n \to \hat{v}_1$ in $C^1([0,1])$ and $g_1(f_n) \to g_1(f) = 0$ weakly in $L^2([0,1])$, and $\hat{v}_1$ is a positive solution to

$$-e^{-R_1(x)}\left(D_1(x)e^{R_1(x)}\hat{v}_1'\right)' = -d_1\hat{v}_1, \qquad \hat{v}_1'(0) = \hat{v}_1'(1) = 0.$$

This implies $d_1 = 0$, a contradiction to our assumption $d_1 \in (0, d_1^*)$.

Next we consider the case $d_2^n \to d_2^*$. We define $\hat{v}_1^n$, $\hat{v}_1^n$ and f_n as above. By the same argument we know that by passing to a subsequence, $\hat{v}_1^n \to \hat{v}_1$ and $\hat{v}_2^n \to \hat{v}_2$ in $C^1([0,1])$, $f_n \to f$ and $g_i(f_n) \to g_i(f)$ in $L^2([0,1])$, and $\hat{v}_2$, $\hat{v}_1$ are positive solutions to

$$-e^{-R_2(x)}\left(D_2(x)e^{R_2(x)}\hat{v}_2'\right) = \left[g_2(f) - d_2^*\right]\hat{v}_2, \qquad \hat{v}_2'(0) = \hat{v}_2'(1) = 0, \tag{2.24}$$

and

$$-e^{-R_1(x)}\left(D_1(x)e^{R_1(x)}\hat{v}_1'\right)' = \left[g_1(f) - d_1\right]\hat{v}_1, \qquad \hat{v}_1'(0) = \hat{v}_1'(1) = 0, \tag{2.25}$$

respectively.

Let us now look at the sequence $\{\|v_1^n\|_\infty\}$. If this sequence is not bounded, then by passing to a subsequence we have $\{\|v_1^n\|_\infty\} \to \infty$ and hence $v_1^n = \|v_1^n\|_\infty \hat{v}_1^n \to \infty$ uniformly in $[0,1]$. This implies that $f \equiv 0$ and (2.25) becomes

$$-e^{-R_1(x)}\left(D_1(x)e^{R_1(x)}\hat{v}_1'\right)' = -d_1\hat{v}_1, \qquad \hat{v}_1'(0) = \hat{v}_1'(1) = 0,$$

which implies $d_1 = 0$, a contradiction. Thus $\{\|v_1^n\|_\infty\}$ is bounded. For the same reason, $\{\|v_2^n\|_\infty\}$ is bounded. So we may assume that

$$\|v_1^n\|_\infty \to \sigma_1 \geqslant 0, \qquad \|v_2^n\|_\infty \to \sigma_2 \geqslant 0.$$

It then follows that

$$f_n(x) \to e^{-k_0 x - \int_0^x (\sigma_1 e^{R_1} \hat{v}_1 + \sigma_2 e^{R_2} \hat{v}_2)\, ds} \quad \text{uniformly in } [0, 1].$$

Thus

$$f(x) = e^{-k_0 x - \int_0^x (\sigma_1 e^{R_1} \hat{v}_1 + \sigma_2 e^{R_2} \hat{v}_2)\, ds} \leqslant e^{-k_0 x}$$

with equality holding if and only if $\sigma_1 = \sigma_2 = 0$. It follows that $g_2(f(x)) \leqslant g_2(e^{-k_0 x})$ with equality holding for all $x \in [0, h]$ if and only if $\sigma_1 = \sigma_2 = 0$. Form this and (2.24) we deduce

$$d_2^* = -\lambda_1^{(2)}\big(g_2(f)\big) \leqslant -\lambda_1^{(2)}\big(-g_2\big(e^{-k_0 x}\big)\big),$$

with equality holding if and only if $\sigma_1 = \sigma_2 = 0$. Thus in view of the definition of d_2^*, we necessarily have $\sigma_1 = \sigma_2 = 0$ and thus $f(x) = e^{-k_0 x}$. We now use (2.25) and find

$$d_1 = -\lambda_1^{(1)}\big(-g_1(f)\big) = -\lambda_1^{(1)}\big(-g_1\big(e^{-k_0 x}\big)\big).$$

That is, $d_1 = d_1^*$, a contradiction to our assumption on d_1. $\square$

On the other hand, we have existence results with more explicit sufficient conditions than (2.9). Let $D_0(x) \in C^1([0, 1])$ and $\alpha_0(x) \in C^1([0, 1])$ be two positive functions. Let D be a positive parameter and α be a nonpositive parameter. Set $D(x) = D D_0(x)$ and $\alpha(x) = \alpha\alpha_0(x)$.

Lemma 2.6. *Let $\pi(x) \in C([0, 1])$ be a strictly increasing function and $\lambda_1(\pi(x))$ be the principal eigenvalue of the eigenvalue problem*

$$-e^{-R(x)}\big(D(x)e^{R(x)}\phi'\big)' + \pi(x)\phi = \lambda\phi, \qquad \phi'(0) = \phi'(1) = 0,$$

where $R(x) = \int_0^x \frac{\alpha(s)}{D(s)}\, ds$. Then

$$\lambda_1\big(\pi(x)\big) \to \pi(0) \quad \text{as } D \to 0; \qquad \lambda_1\big(\pi(x)\big) \to \pi(0) \quad \text{as } \alpha \to -\infty.$$

Proof. The variational formulation of $\lambda_1(\pi(x))$ is

$$\lambda_1\big(\pi(x)\big) = \inf_{\phi \in H^1([0,1])} \frac{\int_0^1 e^{R(x)}[D(x)|\phi'(x)|^2 + \pi(x)\phi^2(x)]\, dx}{\int_0^1 e^{R(x)}\phi^2(x)\, dx}.$$

Clearly

$$\pi(0) = \min_{x \in [0,1]} \pi(x) \leqslant \lambda_1\big(\pi(x)\big) \leqslant \max_{x \in [0,1]} \pi(x) = \pi(1). \tag{2.26}$$

Take

$$\phi(x)=\phi_\epsilon(x)=\begin{cases}1, & 0\leqslant x\leqslant\epsilon,\\ 2-\frac{x}{\epsilon}, & \epsilon<x\leqslant 2\epsilon,\\ 0, & x>2\epsilon.\end{cases}$$

Then

$$\lambda_1\big(\pi(x)\big)\leqslant\frac{\max_{x\in[0,1]}D(x)}{\epsilon^2}\frac{\int_\epsilon^{2\epsilon}e^{R(x)}\,dx}{\int_0^\epsilon e^{R(x)}\,dx}+\pi(2\epsilon). \tag{2.27}$$

Since $R(x)$ is decreasing, we have

$$\int\limits_\epsilon^{2\epsilon}e^{R(x)}\,dx\leqslant\int\limits_0^\epsilon e^{R(x)}\,dx$$

and hence

$$\lambda_1\big(\pi(x)\big)\leqslant\frac{D\max_{x\in[0,1]}D_0(x)}{\epsilon^2}+\pi(2\epsilon).$$

Therefore

$$\limsup_{D\to0}\lambda_1\big(\pi(x)\big)\leqslant\pi(2\epsilon).$$

Letting $\epsilon\to0$, we obtain

$$\limsup_{D\to0}\lambda_1\big(\pi(x)\big)\leqslant\pi(0).$$

Together with (2.26), we obtain

$$\lim_{D\to0}\lambda_1\big(\pi(x)\big)=\pi(0).$$

To prove the second conclusion we look at (2.27) again. Let $K_1=\max_{x\in[0,1]}|\frac{\alpha_0(x)}{D_0(x)}|$ and $K_2=\min_{x\in[0,1]}|\frac{\alpha_0(x)}{D_0(x)}|$. Then

$$\begin{aligned}\lambda_1\big(\pi(x)\big)&\leqslant\frac{\max_{x\in[0,1]}D(x)}{\epsilon^2}\frac{\int_\epsilon^{2\epsilon}e^{K_2\alpha x/D}\,dx}{\int_0^\epsilon e^{K_1\alpha x/D}\,dx}+\pi(2\epsilon)\\ &\leqslant\frac{\max_{x\in[0,1]}D(x)}{\epsilon^2}\frac{K_1}{K_2}\frac{e^{2K_2\alpha\epsilon/D}-e^{K_2\alpha\epsilon/D}}{e^{K_1\alpha\epsilon/D}-1}+\pi(2\epsilon).\end{aligned}$$

It follows that

$$\limsup_{\alpha\to-\infty}\lambda_1\big(\pi(x)\big)\leqslant\pi(2\epsilon).$$

Letting $\epsilon\to0$ we obtain

$$\limsup_{\alpha\to-\infty}\lambda_1\big(\pi(x)\big)\leqslant\pi(0).$$

In view of (2.26) we obtain

$$\lim_{\alpha\to-\infty}\lambda_1\big(\pi(x)\big)=\pi(0). \qquad \square$$

Lemma 2.7. *Let $v^*_{d_i}$ be the unique positive solution of (2.8) for $i=1,2$. Then*

$$\sup_{D_i(x),\alpha_i(x)}\int_0^1 e^{R_i(s)}v^*_{d_i}(s)\,ds\to 0 \quad \text{as } d_i\to g_i(1),\ i=1,2. \tag{2.28}$$

Proof. Multiplying the equation for $v^*_{d_i}$ by $e^{R_i(x)}$ and integrating the resultant equation over $[0,1]$ one readily has

$$\begin{aligned} d_i\int_0^1 e^{R_i(x)}v^*_{d_i}(x)\,dx &= \int_0^1 g_i\big(e^{-k_0x-\int_0^x e^{R_i(s)}v^*_{d_i}(s)\,ds}\big)e^{R_i(x)}v^*_{d_i}(x)\,dx \\ &\leqslant \int_0^1 g_i\big(e^{-\int_0^x e^{R_i(s)}v^*_{d_i}(s)\,ds}\big)e^{R_i(x)}v^*_{d_i}(x)\,dx \\ &= \int_0^{\int_0^1 e^{R_i(x)}v^*_{d_i}(x)\,dx} g_i\big(e^{-\xi}\big)\,d\xi. \end{aligned}$$

Now suppose up to a subsequence

$$\int_0^1 e^{R_i(s)}v^*_{d_i}(s)\,ds\to\kappa_i>0 \quad \text{as } d_i\to g_i(1).$$

We would have

$$g_i(1)\kappa_i\leqslant\int_0^{\kappa_i} g_i\big(e^{-\xi}\big)\,d\xi<g_i(1)\kappa_i,$$

which is impossible. This finishes the proof of the lemma. $\square$

Theorem 2.8. *Let $D_1(x)\in C^1([0,1])$ be fixed and positive, $\alpha_1(x)\in C^1([0,1])$ also be fixed. Fix a $d_1\in(0,-\lambda_1^{(1)}[-g_1(e^{-k_0x})])$. Let $\alpha_2(x)\in C^1([0,1])$ be a fixed nonpositive function. Then there exist two constants $c_1, c_2=c_2(c_1)$ such that whenever $d_2\in(c_1,-\lambda_1^{(1)}[-g_1(e^{-k_0x})])$ and $\sup_{x\in[0,1]}D_2(x)\leqslant c_2$, (2.9) holds, and hence (2.5) has at least one positive solution.*

Proof. Let c_3 be the unique constant such that $d_1 = -\lambda_1^{(1)}[-g_1(e^{-k_0 x - c_3})]$. By Lemma 2.7, we can find a positive constant c_1 such that if $d_2 \in (c_1, g_2(1))$, one has

$$\sup_{D_2(x),\alpha_2(x)} \int_0^1 e^{R_2(x)} v_{d_2}^*(x)\,dx < c_3.$$

Now since $-g_2(e^{-k_0 x - \int_0^x e^{R_1(s)} v_{d_1}^*(s)\,ds})$ is a fixed strictly increasing function, one has by Lemma 2.6,

$$-\lambda_1^{(2)}\left[-g_2\left(e^{-k_0 x - \int_0^x e^{R_1(s)} v_{d_1}^*(s)\,ds}\right)\right] \to g_2(1) \quad \text{as } \max_{x\in[0,1]} D_2(x) \to 0.$$

Thus we can find $c_2 > 0$ sufficiently small such that for $\max_{x\in[0,1]} D_2(x) < c_2$ one has

$$0 < d_2 < -\lambda_1^{(2)}\left[-g_2\left(e^{-k_0 x - \int_0^x e^{R_1(s)} v_{d_1}^*(s)\,ds}\right)\right].$$

On the other hand, we have

$$0 < d_1 = -\lambda_1^{(1)}\left[-g_1\left(e^{-k_0 x - c_3}\right)\right] < -\lambda_1^{(1)}\left[-g_1\left(e^{-k_0 x - \int_0^x e^{R_2(s)} v_{d_2}^*(s)\,ds}\right)\right].$$

The proof is complete. □

Theorem 2.9. *Let $D_1(x) \in C^1([0,1])$ be fixed and positive, $\alpha_1(x) \in C^1([0,1])$ also be fixed. Fix a $d_1 \in (0, -\lambda_1^{(1)}[-g_1(e^{-k_0 x})])$. Let $D_2(x) \in C^1([0,1])$ be a fixed positive function and $\alpha_2(x) = \alpha_2\alpha_0(x)$ with $\alpha_2 < 0$ a parameter and $\alpha_0(x) \in C^1([0,1])$ be a fixed positive function. Then there exist two constants $c_1, c_2 = c_2(c_1)$ such that whenever $d_2 \in (c_1, -\lambda_1^{(1)}[-g_1(e^{-k_0 x})])$ and $\alpha_2 < -c_2$, (2.9) holds, and hence (2.5) has at least one positive solution.*

Proof. Let c_3 be the unique constant such that $d_1 = -\lambda_1^{(1)}[-g_1(e^{-k_0 x - c_3})]$. By Lemma 2.7, we can find a positive constant c_1 such that if $d_2 \in (c_1, g_2(1))$, one has

$$\sup_{D_2(x),\alpha_2(x)} \int_0^1 e^{R_2(x)} v_{d_2}^*(x)\,dx < c_3.$$

Now since $-g_2(e^{-k_0 x - \int_0^x e^{R_1(s)} v_{d_1}^*(s)\,ds})$ is a fixed strictly increasing function, one has by Lemma 2.6,

$$-\lambda_1^{(2)}\left[-g_2\left(e^{-k_0 x - \int_0^x e^{R_1(s)} v_{d_1}^*(s)\,ds}\right)\right] \to g_2(1) \quad \text{as } \alpha_2 \to -\infty.$$

Thus we can find $c_2 > 0$ sufficiently large such that for $\alpha_2 < -c_2$ one has

$$0 < d_2 < -\lambda_1^{(2)}\left[-g_2\left(e^{-k_0 x - \int_0^x e^{R_1(s)} v_{d_1}^*(s)\,ds}\right)\right].$$

On the other hand, we have

$$0 < d_1 = -\lambda_1^{(1)}\left[-g_1\left(e^{-k_0 x - c_3}\right)\right] < -\lambda_1^{(1)}\left[-g_1\left(e^{-k_0 x - \int_0^x e^{R_2(s)} v_{d_2}^*(s)\,ds}\right)\right].$$

The proof is complete. □

3. Existence and nonexistence of positive steady states for 3-species system

In this section, we consider the positive steady-state solutions of (1.8)–(1.10) ($n=3$). That is we study the positive solutions of the system

$$\begin{cases} -D_1 v_1'' - \alpha_1 v_1' = \big[g_1\big(I(x)\big) - d_1\big] v_1, & 0 < x < 1, \\ -D_2 v_2'' - \alpha_2 v_2' = \big[g_2\big(I(x)\big) - d_2\big] v_2, & 0 < x < 1, \\ -D_3 v_3'' - \alpha_3 v_3' = \big[g_3\big(I(x)\big) - d_3\big] v_3, & 0 < x < 1, \\ v_i'(0) = v_i'(1) = 0, & i = 1, 2, 3, \end{cases} \tag{3.1}$$

where

$$I(x) = e^{-k_0 x} \exp\left(- \int_0^x \big[v_1(y) e^{(\alpha_1/D_1)y} + v_2(y) e^{(\alpha_2/D_2)y} + v_3(y) e^{(\alpha_3/D_3)y} \big]\, dy \right). \tag{3.2}$$

We first find a necessary condition for (3.1) to have a positive solution. For that, we denote by $\lambda_1^{(i)}(\Psi)$, $i=1,2,3$ the principal eigenvalue of the eigenvalue problem

$$\begin{cases} -D_i \phi'' - \alpha_i \phi' + \Psi(x)\phi = \lambda \phi, & 0 < x < 1, \\ \phi'(0) = \phi'(1) = 0. \end{cases}$$

It is well known (see [2]) that $\lambda_1^{(i)}(\Psi)$ is a continuous function of Ψ in $C([0,1])$ and $\lambda_1^{(i)}(\Psi_1) \geqslant \lambda_1^{(i)}(\Psi_2)$ for $\Psi_1 \geqslant \Psi_2$ and equality holds only if $\Psi_1 \equiv \Psi_2$.

Define

$$d_i^* = -\lambda_1^{(i)}\big(-g_i\big(e^{-k_0 x}\big)\big), \quad i = 1, 2, 3. \tag{3.3}$$

Then if (3.1) has a positive solution (v_1, v_2, v_3), we have

$$-d_i = -\lambda_1^{(i)}\big(g_i\big(I(x)\big)\big) \in \big(-d_i^*, 0\big).$$

We thus obtain a necessary condition for (3.1) to have a positive solution:

$$d_i \in \big(0, d_i^*\big), \quad i = 1, 2, 3. \tag{3.4}$$

However, for a particular triple $d_i \in (0, d_i^*)$, $i=1,2,3$, (3.1) does not always has a positive solution for all diffusion coefficients D_i, $i=1,2,3$. This point can be seen from the following theorem.

Theorem 3.1.

(i) *If there is an $i \in \{1,2,3\}$ such that $d_i > \int_0^1 g_i(e^{-k_0 x})\, dx$, then there exists a constant $D > 0$, such that if $\min\{D_1, D_2, D_3\} \geqslant D$ then (3.1) has only the trivial solution.*

(ii) *If $d_i \in (0, \int_0^1 g_i(e^{-k_0 x})\, dx]$ for all $i = 1, 2, 3$, then there exists a positive constant D such that if $\min\{D_1, D_2, D_3\} \geqslant D$, (3.1) has no positive solution except possibly when the following exceptional situation occurs: there exists a constant $c \geqslant 0$ such that*

$$c_1 = c_2 = c_3 = c,$$

where c_i is uniquely determined by

$$d_i = \int_0^1 g_i\bigl(e^{-(k_0+c_i)x}\bigr)\,dx. \tag{3.5}$$

Proof. Denote by $(v_{1D}, v_{2D}, v_{3D}) \in C([0,1]) \times C([0,1]) \times C([0,1])$ a positive solution of (3.1) with $D = (D_1, D_2, D_3)$. Suppose there is a sequence of $D = (D_1, D_2, D_3)$, say $D_n = (D_{1n}, D_{2n}, D_{3n})$, such that $\min\{D_{1n}, D_{2n}, D_{3n}\} \to \infty$ and that (3.1) has a positive solution with $D = D_n$. Set $v_{in} = v_{iD_n}$ and $\tilde v_{in} = v_{in}/\|v_{in}\|_\infty$, $i = 1,2,3$. Then we have

$$\begin{cases} -\tilde v''_{1n} - \dfrac{\alpha_1}{D_{1n}}\tilde v'_{1n} = \dfrac{1}{D_{1n}}\bigl[g_1\bigl(I_n(x)\bigr) - d_1\bigr]\tilde v_{1n}, \\ -\tilde v''_{2n} - \dfrac{\alpha_2}{D_{2n}}\tilde v'_{2n} = \dfrac{1}{D_{2n}}\bigl[g_2\bigl(I_n(x)\bigr) - d_2\bigr]\tilde v_{2n}, \\ -\tilde v''_{3n} - \dfrac{\alpha_3}{D_{3n}}\tilde v'_{3n} = \dfrac{1}{D_{3n}}\bigl[g_3\bigl(I_n(x)\bigr) - d_3\bigr]\tilde v_{3n}, \\ \bigl(\tilde v'_{1n}, \tilde v'_{2n}, \tilde v'_{3n}\bigr)(0) = \bigl(\tilde v'_{1n}, \tilde v'_{2n}, \tilde v'_{3n}\bigr)(1) = 0, \end{cases} \tag{3.6}$$

where

$$I_n(x) = e^{-k_0x - \int_0^x [\|v_{1n}\|_\infty \tilde v_{1n}(s)e^{(\alpha_1/D_{1n})s} + \|v_{2n}\|_\infty \tilde v_{2n}(s)e^{(\alpha_2/D_{2n})s} + \|v_{3n}\|_\infty \tilde v_{3n}(s)e^{(\alpha_3/D_{3n})s}]\,ds}. \tag{3.7}$$

The right-hand side of the equations in (3.6) is clearly bounded. By the standard elliptic regularity and Sobolev embedding, subject to a subsequence, $\tilde v_{in} \to v_{i0}$ in $C^1([0,1])$ and

$$v'_{i0}(x) = \lim_{n\to\infty} v'_{in}(x) = \lim_{n\to\infty}\left(-\frac{1}{D_{in}}\int_0^x \bigl[g_i\bigl(I_n(s)\bigr) - d_i\bigr]\tilde v_{in}(s)\,ds - \frac{\alpha_i}{D_{in}}\tilde v_{in}(x)\right) = 0, \quad i = 1,2,3.$$

Hence $\tilde v_{in} \to 1$ in $C^1([0,1])$, $i = 1,2,3$.

Now multiplying the equation for $\tilde v_{in}$ by $e^{(\alpha_i/D_{in})x}$ and integrating over $[0,1]$ we deduce

$$\int_0^1 g_i\bigl(e^{-k_0x - \int_0^x [v_{1n}(s)e^{(\alpha_1/D_{1n})s} + v_{2n}(s)e^{(\alpha_2/D_{2n})s} + v_{3n}(s)e^{(\alpha_3/D_{3n})s}]\,ds}\bigr)\tilde v_{in}(x)e^{(\alpha_i/D_{in})x}\,dx$$
$$= d_i \int_0^1 \tilde v_{in}(x)e^{(\alpha_i/D_{in})x}\,dx. \tag{3.8}$$

From (3.8) we have

$$d_i \int_0^1 \tilde v_{in}(x)e^{(\alpha_i/D_{in})x}\,dx \leqslant \int_0^1 g_i\bigl(e^{-\|v_{in}\|_\infty \tilde v_{in}(s)e^{(\alpha_i/D_{in})s}]\,ds}\bigr)\tilde v_{in}(x)e^{(\alpha_i/D_{in})x}\,dx. \tag{3.9}$$

We claim $\|v_{in}\|_\infty$ is bounded away from ∞ for each $i = 1,2,3$. Otherwise, subject to a subsequence, one can let $n \to \infty$ in (3.9) and deduce $d_i \leqslant 0$, which is impossible. Hence we can assume, subject to a subsequence $\|v_{in}\|_\infty \to \tau_i \in [0,\infty)$ as $n \to \infty$.

Now letting $n \to \infty$ in (3.8) we deduce

$$d_i = \int_0^1 g_i\left(e^{-(k_0+c_i)x}\right) dx, \quad i = 1, 2, 3, \tag{3.10}$$

where $v_{jn} \to \tau_j \in [0, \infty)$, $j = 1, 2, 3$, and $c_i = \tau_1 + \tau_2 + \tau_3$, $i = 1, 2, 3$.

From the above discussion, conclusion (ii) of the theorem follows readily. From (3.10), we have $d_i \leqslant \int_0^1 g_i(e^{-k_0 x})\, dx$ and hence conclusion (i) of the theorem also follows. □

From Theorem 3.1 we find that when the diffusion coefficients are very large, the phytoplankton species cannot coexist generically. To see this point more clearly, we look at the widely used nonlinearity

$$g_i(I) = \frac{m_i I}{\delta_i + I}, \quad i = 1, 2, 3,$$

where m_i, δ_i are positive constants. In this case (3.10) becomes

$$\frac{m_i}{d_i} \ln\left(\frac{\delta_i + 1}{\delta_i + e^{-(k_0+c)}}\right) = k_0 + c, \quad i = 1, 2, 3. \tag{3.11}$$

Clearly (3.11) implies rather severe restrictions on the parameters. For example even under the restriction $\delta_1 = \delta_2 = \delta_3 = \delta$, we still need

$$\frac{m_1}{d_1} = \frac{m_2}{d_2} = \frac{m_3}{d_3}$$

to guarantee (3.11).

Remark 3.2. Theorem 3.1 suggests that competitive exclusion may happen when the diffusions of the system are sufficiently large. Clearly, a result corresponding to Theorem 3.1 holds also for the 2-species system.

In sharp contrast, when the diffusion coefficients of the system are small, we will prove coexistence of the three phytoplankton is possible.

For that, we consider the equation, for each $i \in \{1, 2, 3\}$,

$$\begin{cases} -D_i v'' - \alpha_i v' = \left[g_i\left(e^{-k_0 x - \int_0^x v(s) e^{(\alpha_i/D_i)s}\, ds}\right) - d_i\right] v, & 0 < x < 1, \\ v'(0) = v'(1) = 0. \end{cases} \tag{3.12}$$

From [9] (see [8] for the case $\alpha_i = 0$), (3.12) has a positive solution if and only if $d_i \in (0, d_i^*)$, where d_i^* is defined by (3.3); moreover, the positive solution is unique. We denote by v_{d_i} the positive solution corresponding to $d_i (\in (0, d_i^*))$.

By using topological degree argument, we can prove the following

Theorem 3.3. *Suppose that*

$$\begin{aligned} &0 < d_1 < -\lambda_1^{(1)}\left[-g_1\left(\sigma_{d_2 d_3}(x)\right)\right], \\ &0 < d_2 < -\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_1 d_3}(x)\right)\right], \\ &0 < d_3 < -\lambda_1^{(3)}\left[-g_3\left(\sigma_{d_1 d_2}(x)\right)\right], \end{aligned} \tag{3.13}$$

where

$$\sigma_{d_2d_3}(x) = e^{-k_0x-\int_0^x [e^{(\alpha_2/D_2)y} v_{d_2}(y)+e^{(\alpha_3/D_3)y} v_{d_3}(y)]\,dy}, \quad x \in [0,1],$$
$$\sigma_{d_1d_3}(x) = e^{-k_0x-\int_0^x [e^{(\alpha_1/D_1)y} v_{d_1}(y)+e^{(\alpha_3/D_3)y} v_{d_3}(y)]\,dy}, \quad x \in [0,1],$$
$$\sigma_{d_1d_2}(x) = e^{-k_0x-\int_0^x [e^{(\alpha_1/D_1)y} v_{d_1}(y)+e^{(\alpha_2/D_2)y} v_{d_2}(y)]\,dy}, \quad x \in [0,1].$$

Then (3.1) *has at least one positive solution.*

The proof of this theorem is rather long, we will do it near the end of the paper (see Section 3). Unfortunately condition (3.13) is rather implicit. Our next theorem gives a specific range of parameters for which (3.13) is satisfied.

Theorem 3.4. *Suppose $\alpha_1, \alpha_2, \alpha_3 \in R^1$ and at least two of the three α_i $(i = 1, 2, 3)$ are nonpositive. Then there exist suitable (D_1, D_2, D_3) and (d_1, d_2, d_3) such that* (3.13) *holds, and hence there is at least one positive solution to* (3.1)*.*

The range of (D_1, D_2, D_3) and (d_1, d_2, d_3) where (3.13) holds will become clear in the proof.

Proof of Theorem 3.4. Let α be a nonpositive constant, D be a positive parameter, $\pi(x)$ be a continuous, strictly increasing function on $[0, 1]$. Denote by $\lambda_D(\pi(x))$ the principal eigenvalue of the eigenvalue problem

$$\begin{cases} -D\varphi'' - \alpha\varphi' + \pi(x)\varphi = \lambda\varphi, & 0 < x < 1, \\ \varphi'(0) = \varphi'(1) = 0. \end{cases}$$

By the same technique as used in the proof of Theorem 3.6 of [14], we can prove $\lambda_D(\pi(x))$ is strictly increasing as a function of $D \in (0, \infty)$, moreover,

$$\lim_{D\to 0} \lambda_D\big(\pi(x)\big) = \pi(0), \qquad \lim_{D\to\infty} \lambda_D\big(\pi(x)\big) = \int_0^1 \pi(x)\,dx. \tag{3.14}$$

Consider the equation

$$\begin{cases} -Dv'' - \alpha v' = \big[g\big(e^{-k_0x-\int_0^x v(s)e^{(\alpha/D)s}\,ds}\big) - d\big]v, & 0 < x < 1, \\ v'(0) = v'(1) = 0, \end{cases} \tag{3.15}$$

where g satisfies the conditions for g_i, $i = 1, 2, 3$ in (1.4).

By Theorem 3.1 of [14], $d \in (0, -\lambda_D(-g(e^{-k_0x})))$ is a necessary and sufficient condition for (3.15) to have a positive solution $v_{D,d}$, moreover the positive solution is unique. Clearly if $0 < d < g(e^{-k_0})$, then (3.15) has a unique positive solution for any $D \in (0, \infty)$ and $\alpha \in R^1$.

Multiplying (3.15) by $e^{(\alpha/D)x}$ and integrating the resultant equation over $[0, 1]$, we have

$$\begin{aligned} d\int_0^1 e^{(\alpha/D)x} v(x)\,dx &= \int_0^1 g\big(e^{-k_0x-\int_0^x e^{(\alpha/D)s} v(s)\,ds}\big)e^{(\alpha/D)x} v(x)\,dx \\ &\leqslant \int_0^1 g\big(e^{-\int_0^x e^{(\alpha/D)s} v(s)\,ds}\big)e^{(\alpha/D)x} v(x)\,dx \end{aligned}$$

$$\leqslant \int_0^{\int_0^1 e^{(\alpha/D)x} v(x)\,dx} g\big(e^{-s}\big)\,ds \leqslant C := \int_0^\infty g\big(e^{-s}\big)\,ds. \tag{3.16}$$

Define

$$c_d = \sup_{\alpha\in R^1, D>0} \int_0^1 e^{(\alpha/D)x} v_{D,\alpha}(x)\,dx. \tag{3.17}$$

We show that

$$c_d \to 0 \quad \text{as } d \to g(1). \tag{3.18}$$

Assume (3.18) does not hold. Then there exists $D_n > 0, \alpha_n \in R^1$ and $d_n \to g(1)$ such that

$$\int_0^1 v_n(x) e^{(\alpha_n/D_n)x}\,dx \to I_* > 0.$$

It follows from (3.16) that

$$g(1)I_* \leqslant \int_0^{I_*} g\big(e^{-s}\big)\,ds = g\big(e^{-s_*}\big)I_* \quad \text{for some } s_* \in (0, I_*),$$

which is impossible.

We are now ready to complete the proof of the theorem. Without loss of generality, we may assume $\alpha_2 \leqslant 0$, $\alpha_3 \leqslant 0$. We first fix $D_1 > 0$ and choose d_2 such that

$$0 < d_2 < g_2(1).$$

Let c_{d_2} be a constant given by (3.17), but with (D, α, d, g) in (3.15) replaced by $(D_2, \alpha_2, d_2, g_2)$. Choose

$$d_1 \in \big(0, g_1\big(e^{-k_0 - c_{d_2}}\big)\big).$$

By (3.17) we have

$$0 < d_1 < -\lambda_{D_1}\big[-g_1\big(e^{-k_0 x - \int_0^x v_{D_2,d_2}(s) e^{(\alpha_2/D_2)s}\,ds}\big)\big].$$

Now by (3.14), we have

$$\lim_{D_2\to 0} \lambda_{D_2}\big[-g_2\big(e^{-k_0 x - \int_0^x v_{D_1,d_1}(s) e^{(\alpha_1/D_1)s}\,ds}\big)\big] = -g_2(1).$$

We can choose D_2 sufficiently small such that

$$0 < d_2 < -\lambda_{D_2}\big[-g_2\big(e^{-k_0 x - \int_0^x v_{D_1,d_1}(s) e^{(\alpha_1/D_1)s}\,ds}\big)\big].$$

Choose $\epsilon > 0$ such that

$$0 < d_1 < -\lambda_{D_1}\big[-g_1\big(e^{-k_0x-\int_0^x v_{D_2,d_2}(s)e^{(\alpha_2/D_2)s}\,ds-\epsilon}\big)\big],$$
$$0 < d_2 < -\lambda_{D_2}\big[-g_2\big(e^{-k_0x-\int_0^x v_{D_1,d_1}(s)e^{(\alpha_1/D_1)s}\,ds-\epsilon}\big)\big].$$

By (3.17) and (3.18), we can choose $d_3 > 0$ such that $g_3(1) - d_3 > 0$ is small enough such that

$$\int_0^1 v_{D_3,d_3}(x)e^{(\alpha_3/D_3)x}\,dx \leqslant \epsilon. \tag{3.19}$$

Hence

$$0 < d_1 < -\lambda_{D_1}\big[-g_1\big(e^{-k_0x-\int_0^x v_{D_2,d_2}(s)e^{(\alpha_2/D_2)s}\,ds-\int_0^x v_{D_3,d_3}(s)e^{(\alpha_3/D_3)s}\,ds}\big)\big], \tag{3.20}$$
$$0 < d_2 < -\lambda_{D_2}\big[-g_2\big(e^{-k_0x-\int_0^x v_{D_1,d_1}(s)e^{(\alpha_1/D_1)s}\,ds-\int_0^x v_{D_3,d_3}(s)e^{(\alpha_3/D_3)s}ds}\big)\big]. \tag{3.21}$$

Finally by (3.14) again, we can choose D_3 small enough such that

$$0 < d_3 < -\lambda_{D_3}\big[-g_3\big(e^{-k_0x-\int_0^x v_{D_1,d_1}(s)e^{(\alpha_1/D_1)s}\,ds-\int_0^x v_{D_2,d_2}(s)e^{(\alpha_2/D_2)s}ds}\big)\big].$$

Note that (3.17), (3.19) and hence (3.20) and (3.21) are not affected by the choice of D_3. The proof is complete. □

Theorem 3.4 tells us that if at least two of the three species are buoyant, when the death rates and the turbulence diffusions of the species are sufficiently small, the three species can coexist in the same water column.

Theorem 3.5. *Suppose D_1, D_2, D_3 are fixed. We can choose suitable $\alpha_1, \alpha_2, \alpha_3$ and d_1, d_2, d_3 such that* (3.13) *holds, and hence there exists at least one positive solution to* (3.1)*.*

Proof. Let $\pi(x)$ be any continuous strictly increasing function on $[0, 1]$. Let λ_α be the principal eigenvalue of the eigenvalue problem

$$\begin{cases} -D\varphi'' - \alpha\varphi' + \pi(x)\varphi = \lambda\varphi, & 0 < x < 1, \\ \varphi'(0) = \varphi'(1) = 0. \end{cases}$$

By the same technique as in the proof of Theorem 3.2 of [14], $\lambda_\alpha(\pi(x))$ is strictly increasing as a function of $\alpha \in (-\infty, \infty)$, moreover,

$$\lim_{\alpha\to-\infty} \lambda_\alpha\big(\pi(x)\big) = \pi(0), \qquad \lim_{\alpha\to\infty} \lambda_\alpha\big(\pi(x)\big) = \pi(1). \tag{3.22}$$

We also have the same c_d as in (3.17) with c_d independent of $\alpha \in R^1, D > 0$:

$$c_d = \sup_{\alpha\in R^1, D>0} \int_0^1 e^{(\alpha/D)x} v(x)\,dx$$

and

$$c_d \to 0, \quad \text{as } d \to g(1).$$

We can now begin to choose suitable α_1, α_2, α_3 and d_1, d_2, d_3 such that (3.13) holds. Choose α_2 and d_1 such that $0 < d_1 < g_1(e^{-k_0})$. Let c_{d_1} be a constant chosen according to (3.17). Choose $d_2 \in (0, g_2(e^{-k_0-c_{d_1}}))$. Now we can choose α_1 sufficiently negative to ensure

$$\begin{aligned} 0 < d_1 &< -\lambda_{\alpha_1}\big[-g_1\big(e^{-k_0x-\int_0^x v_{\alpha_2,d_2}(s)e^{(\alpha_2/D_2)s}\,ds}\big)\big], \\ 0 < d_2 &< -\lambda_{\alpha_2}\big[-g_2\big(e^{-k_0x-\int_0^x v_{\alpha_1,d_1}(s)e^{(\alpha_1/D_1)s}\,ds}\big)\big]. \end{aligned}$$

Choose $\epsilon > 0$ such that

$$\begin{aligned} 0 < d_1 &< -\lambda_{\alpha_1}\big[-g_1\big(e^{-k_0x-\int_0^x v_{\alpha_2,d_2}(s)e^{(\alpha_2/D_2)s}\,ds-\epsilon}\big)\big], \\ 0 < d_2 &< -\lambda_{\alpha_2}\big[-g_2\big(e^{-k_0x-\int_0^x v_{\alpha_1,d_1}(s)e^{(\alpha_1/D_1)s}\,ds-\epsilon}\big)\big]. \end{aligned} \tag{3.23}$$

Choose $d_3 > 0$ such that $g_3(1) - d_3 > 0$ is small enough such that

$$\int_0^1 v_{\alpha_3,d_3}(s)e^{(\alpha_3/D_3)s}\,ds \leqslant \epsilon. \tag{3.24}$$

By (3.23) and (3.24) we have

$$\begin{aligned} 0 < d_1 &< -\lambda_{\alpha_1}\big[-g_1\big(e^{-k_0x-\int_0^x v_{\alpha_2,d_2}(s)e^{(\alpha_2/D_2)s}\,ds-\int_0^x v_{\alpha_3,d_3}(s)e^{(\alpha_3/D_3)s}\,ds}\big)\big], \\ 0 < d_2 &< -\lambda_{\alpha_2}\big[-g\big(e^{-k_0x-\int_0^x v_{\alpha_1,d_1}(s)e^{(\alpha_1/D_1)s}\,ds-\int_0^x v_{\alpha_3,d_3}(s)e^{(\alpha_3/D_3)s}\,ds}\big)\big]. \end{aligned} \tag{3.25}$$

Finally choose α_3 negative enough such that

$$0 < d_3 < -\lambda_{\alpha_3}\big[-g\big(e^{-k_0x-\int_0^x v_{\alpha_1,d_1}(s)e^{(\alpha_1/D_1)s}\,ds-\int_0^x v_{\alpha_2,d_2}(s)e^{(\alpha_2/D_2)s}\,ds}\big)\big]. \tag{3.26}$$

Note that (3.25) still holds. The proof is complete. □

Remark 3.6. In Theorem 3.4, we assume there is at least two of the three α_i, $i = 1, 2, 3$, are nonpositive. We guess it is purely technical. It would be interesting to find conditions that do not require the signs of these α_is.

4. The proof of Theorem 3.3

In this section, we use topological degree (fixed point index) theory to prove Theorem 3.3. Similar techniques have been used in [4] and [5] in treating classic competition or predator–prey systems. Here we need some a prior estimates specific to our nonlocal problem.

We begin by proving the following boundedness lemma.

Lemma 4.1. *Let $t \in [0, 1]$. Fix $d = (d_1, d_2, d_3)$, $d_i \in (0, d_i^*)$ $(i = 1, 2, 3)$. Suppose that (v_1, v_2, v_3) is a nonnegative solution of*

$$\begin{cases} -D_1 v_1'' - \alpha_1 v_1' + d_1^* v_1 = t\big[g_1\big(I(x)\big) + d_1^* - d_1\big]v_1, & 0 < x < 1, \\ -D_2 v_2'' - \alpha_2 v_2' + d_2^* v_2 = t\big[g_2\big(I(x)\big) + d_2^* - d_2\big]v_2, & 0 < x < 1, \\ -D_3 v_3'' - \alpha_3 v_3' + d_3^* v_3 = t\big[g_3\big(I(x)\big) + d_3^* - d_3\big]v_3, & 0 < x < 1, \\ \big(v_1', v_2', v_3'\big)(0) = \big(v_1', v_2', v_3'\big)(1) = (0,0,0). \end{cases} \tag{4.1}$$

Then there exists a constant $B = B_d$ dependent on d, but independent of t such that

$$\|v_1\|_\infty + \|v_2\|_\infty + \|v_3\|_\infty \leqslant B_d.$$

Proof. Argue indirectly. Assume (v_{1n}, v_{2n}, v_{3n}) is a sequence of nonnegative solutions to (4.1) with $t = t_n$ satisfying

$$\|v_{1n}\|_\infty + \|v_{2n}\|_\infty + \|v_{3n}\|_\infty \to \infty \quad \text{as } n \to \infty.$$

Without loss of generality, we assume $\|v_{1n}\|_\infty \to \infty$ as $n \to \infty$. Setting $\tilde{v}_{1n} := v_{1n}/\|v_{1n}\|_\infty$, we have

$$\begin{cases} -D_1 \tilde{v}_{1n}'' - \alpha_1 \tilde{v}_{1n}' + d_1^* \tilde{v}_{1n} = t_n\big[g_1\big(I_n(x)\big) + d_1^* - d_1\big]\tilde{v}_{1n}, \\ \tilde{v}_{1n}'(0) = \tilde{v}_{1n}'(1) = 0, \end{cases}$$

where

$$I_n(x) = \exp\left(-k_0 x - \int_0^x \big[e^{(\alpha_1/D_1)y} v_{1n}(y) + e^{(\alpha_2/D_2)y} v_{2n}(y) + e^{(\alpha_3/D_3)y} v_{3n}(y)\big]\,dy\right).$$

Note that $[g_1(I_n(x)) + d_1^* - d_1]\tilde{v}_{1n}$ is bounded in $C([0,1])$ with respect to n. By the standard elliptic regularity and Sobolev embedding theorems, subject to a subsequence, $\tilde{v}_{1n} \to v_0$ in $C^1([0,1])$. We may also assume $g_1(I_n) \to g_0$ weakly in $L^2([0,1])$ and $t_n \to t_0 \in [0,1]$. Then v_0 satisfies

$$\begin{cases} -D_1 v_0'' - \alpha_1 v_0' + d_1^* v_0 = t_0\big[g_0(x) + d_1^* - d_1\big]v_0, \\ v_0'(0) = v_0'(1) = 0, \quad v_0(x) \geqslant 0, \quad \|v_0\|_\infty = 1. \end{cases}$$

By the strong maximum principle we have

$$v_0(x) > 0 \quad \text{in } [0,1],$$

and hence

$$v_{1n}(x) = \|v_{1n}\|_\infty \tilde{v}_{1n}(x) \to \infty \quad \text{uniformly in } [0,1].$$

Thus

$$g_0(x) = \lim_{n\to\infty} g_1\big(e^{-k_0 x - \int_0^x [e^{(\alpha_1/D_1)y} v_{1n}(y) + e^{(\alpha_2/D_2)y} v_{2n}(y) + e^{(\alpha_3/D_3)y} v_{3n}(y)]\,dy}\big) = 0.$$

Therefore

$$\begin{cases} -D_1 v_0'' - \alpha_1 v_0' + d_1^* v_0 = t_0\big[d_1^* - d_1\big]v_0, \\ v_0'(0) = v_0'(1) = 0, \quad v_0(x) > 0, \quad \|v_0\|_\infty = 1. \end{cases}$$

This implies

$$t_0\big[d_1^* - d_1\big] = \lambda_1^{(1)}\big(d_1^*\big) = d_1^*.$$

From $d_1 \in (0, d_1^*)$ we then have

$$t_0 = \frac{d_1^*}{d_1^* - d_1} > 1,$$

a contradiction. The contradiction finishes the proof. $\square$

Let

$$K = \big\{v \in C\big([0,1]\big):\ v(x) \geqslant 0 \text{ in } [0,1]\big\},$$
$$E = C\big([0,1]\big) \times C\big([0,1]\big) \times C\big([0,1]\big) \quad \text{and} \quad C = K \times K \times K.$$

Clearly K is a cone in $C([0,1])$.

Let L_i $(i = 1,2,3)$ be the solution operator of

$$-D_i v'' - \alpha_i v' + d_i^* v = f_i(x) \quad \big(f_i \in C\big([0,1]\big)\big),\ v'(0) = v'(1) = 0;$$

$G_i(d_i, v_1, v_2, v_3)$ $(i = 1,2,3)$ be the operator defined by

$$G_i(d_i, v_1, v_2, v_3) = \big[g_i(I) + d_i^* - d_i\big]v_i.$$

Let

$$\Omega = \big\{(v_1, v_2, v_3) \in C:\ \|v_1\| + \|v_2\| + \|v_3\| \leqslant B_d + 1\big\}.$$

Define the operator $A : \Omega \to E$ by

$$A(v_1, v_2, v_3) = \big(A_1(v_1, v_2, v_3), A_2(v_1, v_2, v_3), A_3(v_1, v_2, v_3)\big) \quad \text{for any } (v_1, v_2, v_3) \in C,$$

where

$$A_i(v_1, v_2, v_3) = L_i \circ G_i(d_i, v_1, v_2, v_3), \quad i = 1,2,3.$$

Then it is easy to see that (v_1, v_2, v_3) solves (3.1) if and only if it is a fixed point of A. Clearly $A : \Omega \to C$ is a completely continuous operator. Its derivative operator at $v = (v_1, v_2, v_3)$ is

$$A_i'(v)h = L_i \circ G_i'(v)h \quad (i = 1,2,3),\ h = (h_1, h_2, h_3)^T,$$

where $(h_1, h_2, h_3)^T$ denotes the transpose of the row matrix (h_1, h_2, h_3) and

$$\begin{cases} G_{iv_j}'(v)h = \delta_{ij}\big(g_i\big(I(x)\big) + d_i^* - d_i\big)h_j - g_i'\big(I(x)\big)I(x)v_i \displaystyle\int_0^x \sum_{j=1}^3 e^{(\alpha_j/D_j)y} h_j(y)\,dy, \\ h_j'(0) = h_j'(1) = 0, \quad i,j = 1,2,3, \end{cases}$$

where $\delta_{ij} = 1$, for $i = j$; $\delta_{ij} = 0$, for $i \neq j$.

For $d_i \in (0, d_i^*)$ $(i = 1, 2, 3)$, we have the following result.

Lemma 4.2.

$$\deg_C(I - A, \Omega, 0) = 1. \tag{4.2}$$

Proof. In fact, for $t \in [0, 1]$, $(v_1, v_2, v_3) = tA(v_1, v_2, v_3)$, $((v_1, v_2, v_3) \in \bar{\Omega})$ is equivalent to

$$\begin{cases} -D_1 v_1'' - \alpha_1 v_1' + d_1^* v_1 = t\big[g_1\big(I(x)\big) + d_1^* - d_1\big]v_1, & 0 < x < 1, \\ -D_2 v_2'' - \alpha_2 v_2' + d_2^* v_2 = t\big[g_2\big(I(x)\big) + d_2^* - d_2\big]v_2, & 0 < x < 1, \\ -D_3 v_3'' - \alpha_3 v_3' + d_3^* v_3 = t\big[g_3\big(I(x)\big) + d_3^* - d_3\big]v_3, & 0 < x < 1, \\ \big(v_1', v_2', v_3'\big)(0) = \big(v_1', v_2', v_3'\big)(1) = (0, 0, 0). \end{cases} \tag{4.3}$$

By Lemma 4.1, there is no nonnegative solution of (4.3) satisfying

$$\|v_1\| + \|v_2\| + \|v_3\| = B_d + 1. \tag{4.4}$$

This implies for any (v_1, v_2, v_3) satisfying (4.4), $t \in [0, 1]$, we have

$$v \neq tAv.$$

By the homotopic invariant property of the topological degree, we have

$$\deg_C(I - A, \Omega, 0) = \deg_C(I, \Omega, 0) = 1. \qquad \square$$

We need a lemma from [4].

Lemma 4.3. *Let E_1, E_2 be ordered Banach spaces with positive cones C_1, C_2, respectively, $E = E_1 \times E_2$ and $C = C_1 \times C_2$. Let D be an open set in C containing 0 and $S_i : \bar{D} \to C_i$ be completely continuous operators, $i = 1, 2$. Denote by (u, v) a general element in C with $u \in C_1$, $v \in C_2$ and $S(u, v) = (S_1(u, v), S_2(u, v))$, $C_2(\varepsilon) = \{v \in C_2\colon \|v\|_{E_2} < \varepsilon\}$. Suppose $U \subset C_1 \cap D$ is relatively open and bounded, and*

$$S_1(u, 0) \neq u \quad \text{for } u \in \partial U, \qquad S_2(u, 0) \equiv 0 \quad \text{for } u \in \bar{U}.$$

Suppose $S_2 : D \to C_2$ extends to a continuously differentiable mapping of a neighborhood of D into E_2, C_2–C_2 is dense in E_2 and $\Sigma = \{u \in U\colon u = S_1(u, 0)\}$. Then the following conclusions are true:

(i) $\deg_C(I - S, U \times C_2(\varepsilon), 0) = 0$ *for $\varepsilon > 0$ small, if for any $u \in \Sigma$, the spectral radius $r(S_2'(u, 0)|_{C_2}) > 1$ and 1 is not an eigenvalue of $S_2'(u, 0)|_{C_2}$ corresponding to a positive eigenvector*;
(ii) $\deg_C(I - S, U \times C_2(\varepsilon), 0) = \deg_{C_1}(I - S_1|_{C_1}, U, 0)$ *for $\varepsilon > 0$ small, if for any $u \in \Sigma$, $r(S_2'(u, 0)|_{C_2}) < 1$.*

By the strong maximum principle, nonnegative solutions of (3.1) can be classified into three classes:

(I) The unique trivial solution $(v_1, v_2, v_3) = (0, 0, 0)$, which exists for all d_1, d_2 and d_3.
(II) Three semitrivial solutions $(v_1, v_2, v_3) = (v_{d_1}, 0, 0)$ for $d_1 \in (0, d_1^*)$, $(v_1, v_2, v_3) = (0, v_{d_2}, 0)$ for $d_2 \in (0, d_2^*)$ and $(v_1, v_2, v_3) = (0, 0, v_{d_3})$ for $d_3 \in (0, d_3^*)$.
(III) Semitrivial solutions of (3.1) which have exactly one component identically zero.

Let $(v_1, 0, 0)$ be the nonnegative solution of (3.1) and ξ be the first eigenvalue of the eigenvalue problem

$$A'_{(v_1,0,0)}(h_1, h_2, h_3)^T = \xi(h_1, h_2, h_3)^T. \tag{4.5}$$

Namely we want to find $(h_1, h_2, h_3) \neq (0, 0, 0)$ such that

$$\begin{cases} -D_1 h_1'' - \alpha_1 h_1' + \left(1 - \xi^{-1}\right) d_1^* h_1 = \xi^{-1}\left[g_1\left(\sigma_1(x)\right) - d_1\right] h_1 \\ -\xi^{-1} g_1'\left(\sigma_1(x)\right)\sigma_1(x) v_1(x) \displaystyle\int_0^x \sum_{j=1}^3 e^{(\alpha_j/D_j)y} h_j(y)\,dy, & x \in (0,1), \\ -D_2 h_2'' - \alpha_2 h_2' + \left(1 - \xi^{-1}\right) d_2^* h_2 = \xi^{-1}\left[g_2\left(\sigma_1(x)\right) - d_2\right] h_2, & x \in (0,1), \\ -D_3 h_3'' - \alpha_3 h_3' + \left(1 - \xi^{-1}\right) d_3^* h_3 = \xi^{-1}\left[g_3\left(\sigma_1(x)\right) - d_3\right] h_3, & x \in (0,1), \\ h_1' = h_2' = h_3' = 0, & x = 0, 1, \end{cases} \tag{4.6}$$

where

$$\sigma_i(x) = e^{-k_0 x - \int_0^x e^{(\alpha_i/D_i)y} v_i(y)\,dy}, \quad i = 1, 2, 3.$$

For later use, we need to calculate ξ. First we find the condition needed for $\xi = 1$. That is we need to solve

$$\begin{cases} -D_1 h_1'' - \alpha_1 h_1' = \left[g_1\left(\sigma_1(x)\right) - d_1\right] h_1 \\ -g_1'\left(\sigma_1(x)\right)\sigma_1(x) v_1(x) \displaystyle\int_0^x \sum_{j=1}^3 e^{(\alpha_j/D_j)y} h_j(y)\,dy, & x \in (0,1), \\ -D_2 h_2'' - \alpha_2 h_2' = \left[g_2\left(\sigma_1(x)\right) - d_2\right] h_2, & x \in (0,1), \\ -D_3 h_3'' - \alpha_3 h_3' = \left[g_3\left(\sigma_1(x)\right) - d_3\right] h_3, & x \in (0,1), \\ h_1' = h_2' = h_3' = 0, & x = 0, 1. \end{cases} \tag{4.7}$$

For this we need the following lemma:

Lemma 4.4. *Let $i \in \{1, 2, 3\}$, $h \in C^2([0, 1])$ satisfy*

$$\begin{cases} -D_i h'' - \alpha_i h' = \left[g_i\left(\sigma_i(x)\right) - d_i\right] h - g_i'\left(\sigma_i(x)\right)\sigma_i(x) v_i(x) \displaystyle\int_0^x e^{(\alpha_i/D_i)y} h(y)\,dy, \\ h'(0) = h'(1) = 0. \end{cases} \tag{4.8}$$

Then $h \equiv 0$.

Proof. Argue indirectly. Assume that $h \not\equiv 0$. We claim $h(0) \neq 0$. Otherwise $h(0) = 0$. Set $\xi = \int_0^x e^{(\alpha_i/D_i)y} h(y)\,dy$, $\eta = h'$. Then (h, ξ, η) is a solution of the linear ODE system

$$\begin{cases} h' = \eta, & 0 < x < 1, \\ \xi' = e^{(\alpha_i/D_i)x} h, & 0 < x < 1, \\ \eta' = D^{-1}\left[-\alpha_i \eta - \left[g_i\left(\sigma_i(x)\right) - d_i\right] h + g_i'\left(\sigma_i(x)\right)\sigma_i(x) v_i(x)\xi\right], & 0 < x < 1, \\ (h(0), \xi(0), \eta(0)) = (0, 0, 0). \end{cases}$$

Clearly $(h,\xi,\eta)\equiv(0,0,0)$ is the unique solution of the system. This contradicts our assumption that $h\not\equiv 0$. This proves our claim that $h(0)\neq 0$. Without loss of generality, we assume $h(0)>0$. We next claim that $h(x)$ changes sign in $(0,1)$. Otherwise, $h(x)>0$ in $[0,1)$. Multiplying the first equation in (4.8) by $e^{(\alpha_i/D_i)x}v_i(x)$ and integrating it over $[0,1]$, we obtain

$$\int_0^1 g_i'\big(\sigma_i(x)\big)\sigma_i(x)v_i^2(x)e^{(\alpha_i/D_i)x}\int_0^x e^{(\alpha_i/D_i)y}h(y)\,dy\,dx=0.$$

This is impossible since the integrand function is clearly nonnegative and not identically zero in $[0,1]$ and henceforth the integral should be positive. Hence $h(x)$ changes sign in $(0,1)$. Let x_0 be the first zero of h. Then $h(x_0)=0$, $h(x)>0$ for $x\in[0,x_0)$. We want to reach a contradiction.

Consider now the eigenvalue problem

$$-D_i\phi''-\alpha_i\phi'=\big[g_i\big(\sigma_i(x)\big)-d_i\big]\phi+\lambda\phi\quad\text{in }(0,x_0),\ \phi'(0)=\phi(x_0)=0. \tag{4.9}$$

We show the first eigenvalue λ_1 of this problem is positive. Let ϕ_1 be a positive eigenfunction corresponding to λ_1. Multiplying the first equation in (4.9) (with $\lambda=\lambda_1$, $\phi=\phi_1$) by $e^{(\alpha_i/D_i)x}v_i$ and integrating it over $[0,x_0]$, we obtain

$$\begin{aligned}
&\int_0^{x_0}\big[g_i(\sigma_i)-d_i\big]e^{(\alpha_i/D_i)x}v_i\phi_1+\lambda_1\int_0^{x_0}e^{(\alpha_i/D_i)x}v_i\phi_1\\
&\quad=-D_i\int_0^{x_0}e^{(\alpha_i/D_i)x}v_i\phi_1''-\alpha_i\int_0^{x_0}e^{(\alpha_i/D_i)x}v_i\phi_1'\\
&\quad=-D_ie^{(\alpha_i/D_i)x}v_i(x)\phi_1'(x)\big|_0^{x_0}+D_i\int_0^{x_0}e^{(\alpha_i/D_i)x}v_i'\phi_1'\\
&\quad=-D_ie^{(\alpha_i/D_i)x_0}v_i(x_0)\phi_1'(x_0)-D_i\int_0^{x_0}e^{(\alpha_i/D_i)x}v_i''\phi_1-\alpha_i\int_0^{x_0}e^{(\alpha_i/D_i)x}v_i'\phi_1\\
&\quad>-D_i\int_0^{x_0}e^{(\alpha_i/D_i)x}v_i''\phi_1-\alpha_i\int_0^{x_0}e^{(\alpha_i/D_i)x}v_i'\phi_1\\
&\quad=\int_0^{x_0}\big[g_i(\sigma_i)-d_i\big]e^{(\alpha_i/D_i)x}v_i\phi_1,
\end{aligned}$$

since by Hopf's lemma we have $\phi_1'(x_0)<0$. It follows that $\lambda_1\int_0^{x_0}e^{(\alpha_i/D_i)x}v_i\phi_1>0$. Hence $\lambda_1>0$.

Now multiply the first equation in (4.9) (with $\phi=\phi_1,\lambda=\lambda_1$) by $e^{(\alpha_i/D_i)x}h$ and integrate over $[0,x_0]$. It follows from the equation for h and the boundary conditions for h and ϕ_1 that

$$\int_0^{x_0}\big[g_i(\sigma_i)-d_i\big]e^{(\alpha_i/D_i)x}h\phi_1+\lambda_1\int_0^{x_0}e^{(\alpha_i/D_i)x}h\phi_1$$

$$
\begin{aligned}
&= -D_i \int_0^{x_0} e^{(\alpha_i/D_i)x} h\phi_1'' - \alpha_i \int_0^{x_0} e^{(\alpha_i/D_i)x} h\phi_1' \\
&= -D_i \int_0^{x_0} e^{(\alpha_i/D_i)x} h''\phi_1 - \alpha_i \int_0^{x_0} e^{(\alpha_i/D_i)x} h'\phi_1 + D_i e^{(\alpha_i/D_i)x}\big[h'(x)\phi_1(x) - h(x)\phi_1'(x)\big]\big|_0^{x_0} \\
&= -D_i \int_0^{x_0} e^{(\alpha_i/D_i)x} h''\phi_1 - \alpha_i \int_0^{x_0} e^{(\alpha_i/D_i)x} h'\phi_1 \\
&= \int_0^{x_0} \big[g_i(\sigma_i) - d_i\big] e^{(\alpha_i/D_i)x} h\phi_1 - \int_0^{x_0} \left[g_i'\big(\sigma_i(x)\big)\sigma_i(x) v_i(x) \int_0^x e^{(\alpha_i/D_i)y} h(y)\, dy \right] e^{(\alpha_i/D_i)x}\phi_1\, dx.
\end{aligned}
$$

Consequently,

$$
\lambda_1 \int_0^{x_0} e^{(\alpha_i/D_i)x} h(x)\phi_1(x)\, dx = -\int_0^{x_0} \left[g_i'\big(\sigma_i(x)\big)\sigma_i(x) v_i(x) \int_0^x e^{(\alpha_i/D_i)y} h(y)\, dy \right] e^{(\alpha_i/D_i)x}\phi_1(x)\, dx.
$$

This is impossible since the left-hand side of this identity is positive while the right-hand side is at least nonpositive. This contradiction completes the proof of the lemma. □

Denote by v_{d_i} $(0 < d_i < d_i^*,\ i = 1, 2, 3)$ the unique positive solution of the problem

$$
\begin{cases}
-D_i v'' - \alpha_i v' = \big[g_i\big(e^{-k_0 x - \int_0^x e^{(\alpha_i/D_i)y} v(y)\, dy}\big) - d_i\big] v, & 0 < x < 1, \\
v'(0) = v'(1) = 0.
\end{cases}
\tag{4.10}
$$

Let

$$
\sigma_{d_i}(x) = e^{-k_0 x - \int_0^x e^{(\alpha_i/D_i)y} v_{d_i}(y)\, dy}, \quad i = 1, 2, 3.
$$

Then it is clear that

$$
\begin{cases}
0 < -\lambda_1^{(i)}\big[-g_i\big(\sigma_{d_1}(x)\big)\big] < -\lambda_1^{(i)}\big[-g_i\big(e^{-k_0 x}\big)\big] = d_i^* & (i = 2, 3), \\
0 < -\lambda_1^{(j)}\big[-g_j\big(\sigma_{d_2}(x)\big)\big] < -\lambda_1^{(j)}\big[-g_j\big(e^{-k_0 x}\big)\big] = d_j^* & (j = 1, 3), \\
0 < -\lambda_1^{(k)}\big[-g_k\big(\sigma_{d_3}(x)\big)\big] < -\lambda_1^{(k)}\big[-g_k\big(e^{-k_0 x}\big)\big] = d_k^* & (k = 1, 2).
\end{cases}
$$

Lemma 4.5. *Suppose $d_1 \in (0, d_1^*)$, $d_2 \in (0, d_2^*)$, $d_3 \in (0, d_3^*)$,*

$$
\begin{aligned}
&d_i \neq -\lambda_1^{(i)}\big[-g_i\big(\sigma_{d_1}(x)\big)\big] \quad (i = 2, 3), \\
&d_j \neq -\lambda_1^{(j)}\big[-g_j\big(\sigma_{d_2}(x)\big)\big] \quad (j = 1, 3), \\
&d_k \neq -\lambda_1^{(k)}\big[-g_k\big(\sigma_{d_3}(x)\big)\big] \quad (k = 1, 2).
\end{aligned}
$$

Then $(v_{d_1}, 0, 0)$, $(0, v_{d_2}, 0)$, $(0, 0, v_{d_3})$ are all isolated solutions of (3.1) and

$$\operatorname{index}_C\big(A,(v_{d_1},0,0)\big)=\begin{cases}0, & \textit{if } d_2\in(0,-\lambda_1^{(2)}[-g_2(\sigma_{d_1}(x))]) \textit{ or } d_3\in(0,-\lambda_1^{(3)}[-g_3(\sigma_{d_1}(x))]),\\ 1, & \textit{if } d_2\in(-\lambda_1^{(2)}[-g_2(\sigma_{d_1}(x))],d_2^*) \textit{ and } d_3\in(-\lambda_1^{(3)}[-g_3(\sigma_{d_1}(x))],d_3^*);\end{cases}$$

$$\operatorname{index}_C\big(A,(0,v_{d_2},0)\big)=\begin{cases}0, & \textit{if } d_1\in(0,-\lambda_1^{(1)}[-g_1(\sigma_{d_2}(x))]) \textit{ or } d_3\in(0,-\lambda_1^{(3)}[-g_3(\sigma_{d_2}(x))]),\\ 1, & \textit{if } d_1\in(-\lambda_1^{(1)}[-g_1(\sigma_{d_2}(x))],d_1^*) \textit{ and } d_3\in(-\lambda_1^{(3)}[-g_3(\sigma_{d_2}(x))],d_3^*);\end{cases}$$

$$\operatorname{index}_C\big(A,(0,0,v_{d_3})\big)=\begin{cases}0, & \textit{if } d_1\in(0,-\lambda_1^{(1)}[-g_1(\sigma_{d_3}(x))]) \textit{ or } d_2\in(0,-\lambda_1^{(2)}[-g_2(\sigma_{d_3}(x))]),\\ 1, & \textit{if } d_1\in(-\lambda_1^{(1)}[-g_1(\sigma_{d_3}(x))],d_1^*) \textit{ and } d_2\in(-\lambda_1^{(2)}[-g_2(\sigma_{d_3}(x))],d_2^*).\end{cases}$$

Proof. We only prove the conclusion holds for $(v_{d_1},0,0)$. The proofs for $(0,v_{d_2},0)$ and $(0,0,v_{d_3})$ are similar. Suppose that $(v_{d_1},0,0)$ is not an isolated solution, then there exists a sequence $\{(v_{1n},v_{2n},v_{3n})\}$ of nonnegative solutions for (3.1), such that $(v_{1n},v_{2n},v_{3n})\to(v_{d_1},0,0)$ as $n\to\infty$. As $(v_{d_1},0,0)$ is the unique type II solution of (3.1) with the second and the third components identically zero, we may assume that $\|v_{2n}\|_\infty>0$ (or $\|v_{3n}\|_\infty>0$).

Set $\tilde v_{2n}=\frac{v_{2n}}{\|v_{2n}\|_\infty}$. We then have

$$\begin{cases}-D_2\tilde v_{2n}''-\alpha_2\tilde v_{2n}'=\big[g_2\big(I_n(x)\big)-d_2\big]\tilde v_{2n}, & 0<x<1,\\ \tilde v_{2n}'(0)=\tilde v_{2n}'(1)=0, \quad 0<\tilde v_{2n}\leqslant 1, & \|\tilde v_{2n}\|_\infty=1,\end{cases}\tag{4.11}$$

where

$$I_n(x)=\exp\left(-k_0x-\int_0^x\big[e^{(\alpha_1/D_1)y}v_{1n}(y)+e^{(\alpha_2/D_2y}v_{2n}(y)+e^{(\alpha_3/D_3)y}v_{3n}(y)\big]\,dy\right).$$

The right-hand side of (4.11) is bounded. By the standard L^p theory of elliptic regularity and Sobolev embedding theorems, we may assume, by passing to a subsequence, $\tilde v_{2n}\to v_0$ in $C^1([0,1])$. Moreover, v_0 satisfies

$$\begin{cases}-D_2v_0''-\alpha_2v_0'=\big[g_2\big(e^{-k_0x-\int_0^x e^{(\alpha_1/D_1)y}v_{d_1}(y)\,dy}\big)-d_2\big]v_0, & 0<x<1,\\ v_0'(0)=v_0'(1)=0, \quad 0\leqslant v_0\leqslant 1, & \|v_0\|_\infty=1.\end{cases}$$

By the strong maximum principle $v_0>0$ in $[0,1]$. Hence $d_2=-\lambda_1^{(2)}[-g_2(\sigma_{d_1}(x))]$, a contradiction. Thus $(v_{d_1},0,0)$ is an isolated solution.

By Lemma 4.4, it is easy to check that

$$\begin{cases}-D_2h_2''-\alpha_2h_2'+\big(1-\xi^{-1}\big)d_2^*h_2=\xi^{-1}\big[g_2\big(\sigma_1(x)\big)-d_2\big]h_2, & x\in(0,1),\\ -D_3h_3''-\alpha_3h_3'+\big(1-\xi^{-1}\big)d_3^*h_3=\xi^{-1}\big[g_3\big(\sigma_1(x)\big)-d_3\big]h_3, & x\in(0,1),\\ h_2'=h_3'=0, & x=0,1\end{cases}\tag{4.12}$$

has an eigenvalue $\xi>1$ if and only if

$$d_2\in\big(0,-\lambda_1^{(2)}\big[-g_2\big(\sigma_{d_1}(x)\big)\big]\big)\quad\text{or}\quad d_3\in\big(0,-\lambda_1^{(3)}\big[-g_3\big(\sigma_{d_1}(x)\big)\big]\big);$$

and (4.12) has the eigenvalue $\xi<1$ if and only if

$$d_2\in\big(-\lambda_1^{(2)}\big[-g_2\big(\sigma_{d_1}(x)\big)\big],d_2^*\big)\quad\text{and}\quad d_3\in\big(-\lambda_1^{(3)}\big[-g_3\big(\sigma_{d_1}(x)\big)\big],d_3^*\big).$$

Now a use of Theorem 1 in [3] completes the proof of this lemma. $\square$

Next we consider semitrivial solutions of (3.1) which have exactly one component identically zero. For $i=1,2,3$, we denote by T_i the set of semitrivial solutions which have the ith component zero and the other two components positive.

For simplicity, we also write $w_0=(0,0,0)$, $w_1=(v_{d_1},0,0)$, $w_2=(0,v_{d_2},0)$, $w_3=(0,0,v_{d_3})$. Then evidently the set

$$M=\left(\bigcup_{i=1}^{3}T_i\right)\cup\left(\bigcup_{i=0}^{3}\{w_i\}\right)$$

contains all the nonnegative solutions of (3.1) which are not positive.

Lemma 4.6. *T_1 is a compact set in C if w_2 and w_3 are both isolated in Ω; T_2 is a compact set in C if w_3 and w_1 are both isolated in Ω; T_3 is a compact set in C if w_1 and w_2 are both isolated in Ω.*

Proof. We only give the proof for T_3, the cases for T_1 and T_2 are similar.

We first prove that $w_0=(0,0,0)$ is isolated. Otherwise, let $\{(v_{1n},v_{2n},v_{3n})\}$ be a sequence of nonnegative solutions of (3.1) that converges to $(0,0,0)$. Then, by choosing a subsequence, we may assume that $v_{1n}>0$ for all $n=1,2,\ldots$. Set $\tilde v_{1n}=v_{1n}/\|v_{1n}\|_\infty$. Then $\tilde v_{1n}$ satisfies

$$\begin{cases}-D_1\tilde v_{1n}''-\alpha_1\tilde v_{1n}'=\left[g_1\left(I_n(x)\right)-d_1\right]\tilde v_{1n},\\ \tilde v_{1n}'(0)=\tilde v_{1n}'(1)=0,\end{cases}\tag{4.13}$$

where

$$I_n(x)=e^{-k_0x}\exp\left(-\int_0^x\left[e^{(\alpha_1/D_1)y}v_{1n}(y)+e^{(\alpha_2/D_2)y}v_{2n}(y)+e^{(\alpha_3/D_3)y}v_{3n}(y)\right]dy\right).$$

Clearly $g_1(I_n(x))\to g_1(e^{-k_0x})$ in $L^2([0,1])$. The right-hand side of (4.13) is bounded. Thus by elliptic regularity we may assume, by passing to a subsequence, that $\tilde v_{1n}\to v_0$ in $C^1([0,1])$. Moreover v_0 satisfies

$$\begin{cases}-D_1v_0''-\alpha_1v_0'=\left[g_1\left(e^{-k_0x}\right)-d_1\right]v_0,\\ v_0'(0)=v_0'(1)=0,\quad 0\leqslant v_0\leqslant 1,\quad \|v_0\|_\infty=1.\end{cases}$$

By the strong maximum principle, we have $v_0(x)>0$ for $x\in[0,1]$. This implies

$$d_1=-\lambda_1^{(1)}\left[-g_1\left(e^{-k_0x}\right)\right]=d_1^*,$$

contradicting to $d_1\in(0,d_1^*)$. Hence $w_0=(0,0,0)$ is isolated.

Now suppose w_1 and w_2 are also isolated. Let $(v_{1n},v_{2n},0)\in T_3$. By Lemma 4.1, $\{(v_{1n},v_{2n},0)\in T_3\}$ is precompact. Subject to a subsequence, we may assume $(v_{1n},v_{2n},0)\to(v_1,v_2,0)$. Now that w_0,w_1,w_2 are all isolated, hence $v_1\not\equiv 0$, $v_2\not\equiv 0$. By the strong maximum principle $v_1>0$, $v_2>0$. Hence $(v_1,v_2,0)\in T_3$. Thus T_3 is compact. □

Let $E_1=C([0,1])\times C([0,1])$, $E_2=C([0,1])$ and $E=E_1\times E_2$, $C_1=K\times K$, $C_2=K$ and $C=C_1\times C_2$. Then E is an ordered Banach space with positive cone C.

Define $S:\Omega\to C$ by

$$\begin{aligned}S\big((v_1,v_2),v_3\big)&=\big(S_1\big((v_1,v_2),v_3\big),S_2\big((v_1,v_2),v_3\big)\big)\\&=\big(\big(A_1(v_1,v_2,v_3),A_2(v_1,v_2,v_3)\big),A_3(v_1,v_2,v_3)\big).\end{aligned}$$

In order to use Lemma 4.3, we choose a neighborhood $U \subset C_1 \cap \Omega$ of $T_3 \cap C_1$ such that $(v_{d_1}, 0), (0, v_{d_2}) \notin \bar{U}$. Now $S_1(v_1, v_2, 0) = (v_1, v_2)$ with $(v_1, v_2) \in \bar{U}$ if and only if $(v_1, v_2, 0) \in T_3$.

Fix $d_1 \in (0, d_1^*)$, $d_1 \neq -\lambda_1^{(1)}[-g_1(\sigma_{d_i}(x))]$ $(i = 2, 3)$; $d_2 \in (0, d_2^*)$, $d_2 \neq -\lambda_1^{(2)}[-g_2(\sigma_{d_j}(x))]$ $(j = 1, 3)$ and $d_3 \in (0, d_3^*)$, $d_3 \neq -\lambda_1^{(3)}[-g_3(\sigma_{d_k}(x))]$ $(k = 1, 2)$, then we have

Lemma 4.7.

$$\deg_{C_1}(I - S_1|_{C_1}, U, 0) = \begin{cases} 1, & \textit{if } d_1 \in (0, -\lambda_1^{(1)}[-g_1(\sigma_{d_2}(x))]) \textit{ and } d_2 \in (0, -\lambda_1^{(2)}[-g_2(\sigma_{d_1}(x))]), \\ -1, & \textit{if } d_1 \in (-\lambda_1^{(1)}[-g_1(\sigma_{d_2}(x))], d_1^*) \textit{ and } d_2 \in (-\lambda_1^{(2)}[-g_2(\sigma_{d_1}(x))], d_2^*), \\ 0, & \textit{if } [d_1 + \lambda_1^{(1)}[-g_1(\sigma_{d_2}(x))]] \cdot [d_2 + \lambda_1^{(2)}[-g_2(\sigma_{d_1}(x))]] < 0. \end{cases}$$

Proof. By Theorem 2.1 in [9], $(0, 0)$, $(v_{d_1}, 0)$ and $(0, v_{d_2})$ are the only semitrivial solutions to the equation in C_1:

$$\begin{cases} -D_1 v_1'' - \alpha_1 v_1' = \big[g_1\big(\sigma_{12}(x)\big) - d_1\big] v_1, & 0 < x < 1, \\ -D_2 v_2'' - \alpha_2 v_2' = \big[g_2\big(\sigma_{12}(x)\big) - d_2\big] v_2, & 0 < x < 1, \\ v_i'(0) = v_i'(1) = 0, & i = 1, 2, \end{cases} \tag{4.14}$$

where $\sigma_{12}(x) = \exp(-k_0 x - \int_0^x [e^{(\alpha_1/D_1)y} v_1(y) + e^{(\alpha_2/D_2)y} v_2(y)]\, dy)$. Now choose neighborhoods U_0, U_1, U_2 of $(0, 0)$, $(v_{d_1}, 0)$, $(0, v_{d_2})$ in C_1 respectively, such that $\bar{U}_0$, $\bar{U}_1$, $\bar{U}_2$ and $\bar{U}$ are disjoint. It is clear that (4.14) does not have any nonnegative solutions in $\bar{\Omega} \cap C_1$. Therefore we have

$$\deg_{C_1}(I - S_1|_{C_1}, \Omega \cap C_1, 0) = \deg_{C_1}(I - S_1|_{C_1}, U, 0) + \sum_{i=0}^{2} \deg_{C_1}(I - S_1|_{C_1}, U_i, 0).$$

It is clear that $\deg_{C_1}(I - S_1|_{C_1}, U_0, 0) = 0$. By Lemma 4.2, $\deg_{C_1}(I - S_1|_{C_1}, \Omega \cap C_1, 0) = 1$. We thus have

$$\deg_{C_1}(I - S_1|_{C_1}, U, 0) = 1 - \sum_{i=1}^{2} \deg_{C_1}(I - S_1|_{C_1}, U_i, 0).$$

A use of Lemma 4.5 then finishes the proof. □

We call this degree the face index of T_3 and denote it by $\operatorname{index}_f(A, T_3)$. This is well defined because this degree does not depend on the particular choice of the neighborhood U. Thus we have

$$\operatorname{index}_f(A, T_3) = \begin{cases} 1, & \text{if } d_1 \in (0, -\lambda_1^{(1)}[-g_1(\sigma_{d_2}(x))]) \text{ and } d_2 \in (0, -\lambda_1^{(2)}[-g_2(\sigma_{d_1}(x))]), \\ -1, & \text{if } d_1 \in (-\lambda_1^{(1)}[-g_1(\sigma_{d_2}(x))], d_1^*) \text{ and } d_2 \in (-\lambda_1^{(2)}[-g_2(\sigma_{d_1}(x))], d_2^*), \\ 0, & \text{if } [d_1 + \lambda_1^{(1)}[-g_1(\sigma_{d_2}(x))]] \cdot [d_2 + \lambda_1^{(2)}[-g_2(\sigma_{d_1}(x))]] < 0. \end{cases} \tag{4.15}$$

Similarly, we have the results for T_1 and T_2:

$$\operatorname{index}_f(A, T_1) = \begin{cases} 1, & \text{if } d_2 \in (0, -\lambda_1^{(2)}[-g_2(\sigma_{d_3}(x))]) \text{ and } d_3 \in (0, -\lambda_1^{(3)}[-g_3(\sigma_{d_2}(x))]), \\ -1, & \text{if } d_2 \in (-\lambda_1^{(2)}[-g_2(\sigma_{d_3}(x))], d_2^*) \text{ and } d_3 \in (-\lambda_1^{(3)}[-g_3(\sigma_{d_2}(x))], d_3^*), \\ 0, & \text{if } [d_2 + \lambda_1^{(2)}[-g_2(\sigma_{d_3}(x))]] \cdot [d_3 + \lambda_1^{(3)}[-g_3(\sigma_{d_2}(x))]] < 0, \end{cases} \tag{4.16}$$

$$\operatorname{index}_f(A, T_2) = \begin{cases} 1, & \text{if } d_1 \in (0, -\lambda_1^{(1)}[-g_1(\sigma_{d_3}(x))]) \text{ and } d_3 \in (0, -\lambda_1^{(3)}[-g_3(\sigma_{d_1}(x))]), \\ -1, & \text{if } d_1 \in (-\lambda_1^{(1)}[-g_1(\sigma_{d_3}(x))], d_1^*) \text{ and } d_3 \in (-\lambda_1^{(3)}[-g_3(\sigma_{d_1}(x))], d_3^*), \\ 0, & \text{if } [d_1 + \lambda_1^{(1)}[-g_1(\sigma_{d_3}(x))]] \cdot [d_3 + \lambda_1^{(3)}[-g_3(\sigma_{d_1}(x))]] < 0. \end{cases} \tag{4.17}$$

For any $(\bar{v}_1, \bar{v}_2, 0) \in T_3$, we can easily show that $r(S_2'(\bar{v}_1, \bar{v}_2, 0)|_{C_2}) > 1$ if and only if $d_3 < -\lambda_1^{(3)}[-g_3(\bar{\sigma}(x))]$, where

$$\bar{\sigma}(x) = e^{-k_0 x - \int_0^x [e^{(\alpha_1/D_1)y} \bar{v}_1(y) + e^{(\alpha_2/D_2)y} \bar{v}_2(y)]\,dy}, \quad x \in [0, 1],$$

and $r(A_2'(\bar{v}_1, \bar{v}_2, 0)|_{C_2}) < 1$ if and only if $d_3 > -\lambda_1^{(3)}[-g_3(\bar{\sigma}(x))]$. Therefore, by Lemma 4.3, we have

$$\deg_C\big(I - A, U \times C_2(\varepsilon), 0\big) = \begin{cases} 0, & \text{if } d_3 < -\lambda_1^{(3)}[-g_3\big(\bar{\sigma}(x)\big)], \text{ for any } (\bar{v}_1, \bar{v}_2, 0) \in T_3, \\ \operatorname{index}_f(A, T_3), & \text{if } d_3 > -\lambda_1^{(3)}[-g_3\big(\bar{\sigma}(x)\big)], \text{ for any } (\bar{v}_1, \bar{v}_2, 0) \in T_3. \end{cases}$$

Since the above degree does not depend on the particular choices of U and ε, we call this degree the index of T_3 and denote it by $\operatorname{index}_C(A, T_3)$. We can define $\operatorname{index}_C(A, T_1)$ and $\operatorname{index}_C(A, T_2)$ similarly. So we have

$$\operatorname{index}_C(A, T_3) = \begin{cases} 0, & \text{if } d_3 < -\lambda_1^{(3)}[-g_3\big(\bar{\sigma}(x)\big)], \text{ for any } (\bar{v}_1, \bar{v}_2, 0) \in T_3, \\ \operatorname{index}_f(A, T_3), & \text{if } d_3 > -\lambda_1^{(3)}[-g_3\big(\bar{\sigma}(x)\big)], \text{ for any } (\bar{v}_1, \bar{v}_2, 0) \in T_3; \end{cases} \tag{4.18}$$

$$\operatorname{index}_C(A, T_2) = \begin{cases} 0, & \text{if } d_2 < -\lambda_1^{(2)}[-g_2\big(\hat{\sigma}(x)\big)], \text{ for any } (\hat{v}_1, 0, \hat{v}_3) \in T_2, \\ \operatorname{index}_f(A, T_2), & \text{if } d_2 > -\lambda_1^{(2)}[-g_2\big(\hat{\sigma}(x)\big)], \text{ for any } (\hat{v}_1, 0, \hat{v}_3) \in T_2; \end{cases} \tag{4.19}$$

$$\operatorname{index}_C(A, T_1) = \begin{cases} 0, & \text{if } d_1 < -\lambda_1^{(1)}[-g_1\big(\tilde{\sigma}(x)\big)], \text{ for any } (0, \tilde{v}_2, \tilde{v}_3) \in T_1, \\ \operatorname{index}_f(A, T_1), & \text{if } d_1 > -\lambda_1^{(1)}[-g_1\big(\tilde{\sigma}(x)\big)], \text{ for any } (0, \tilde{v}_2, \tilde{v}_3) \in T_1, \end{cases} \tag{4.20}$$

where

$$\hat{\sigma}(x) = e^{-k_0 x - \int_0^x [e^{(\alpha_1/D_1)y} \hat{v}_1(y) + e^{(\alpha_3/D_3)y} \hat{v}_3(y)]\,dy},$$
$$\tilde{\sigma}(x) = e^{-k_0 x - \int_0^x [e^{(\alpha_2/D_2)y} \tilde{v}_2(y) + e^{(\alpha_3/D_3)y} \tilde{v}_3(y)]\,dy}.$$

Lemma 4.8. *Suppose* $\operatorname{index}_C(A, w_i)$ *and* $\operatorname{index}_C(A, T_i)$ $(i = 1, 2, 3)$ *are well defined, and*

$$\sum_{i=1}^{3} \operatorname{index}_C(A, w_i) + \sum_{i=1}^{3} \operatorname{index}_C(A, T_i) \neq 1.$$

Then (3.1) *has at least one positive solution.*

Proof. Suppose that (3.1) has no positive solution and $\operatorname{index}_C(A, w_i)$ $(i = 0, 1, 2, 3)$ and $\operatorname{index}_C(A, T_j)$ $(j = 1, 2, 3)$ are well defined. Then by the additivity of the degree

$$\deg_C(I - A, \Omega, 0) = \sum_{i=0}^{3} \operatorname{index}_C(A, w_i) + \sum_{j=1}^{3} \operatorname{index}_C(A, T_j).$$

It follows from Lemma 4.2 and $\operatorname{index}_C(A, (0, 0, 0)) = 0$ that

$$1 = \sum_{i=1}^{3} \operatorname{index}_C(A, w_i) + \sum_{i=1}^{3} \operatorname{index}_C(A, T_i).$$

This leads to a contradiction. □

Before going further, we prove an important comparison lemma.

Lemma 4.9. *Suppose that v_{d_i} $(0 < d_i < d_i^*, i = 1, 2, 3)$ is the unique positive solution of (4.10). Then for any $(v_1, v_2, v_3) \in T_1 \cup T_2 \cup T_3$, we have*

$$\int_0^x e^{(\alpha_i/D_i)y} v_i(y)\,dy \leqslant \int_0^x e^{(\alpha_i/D_i)y} v_{d_i}(y)\,dy \quad \text{for any } x \in [0, 1],\ i = 1, 2, 3. \tag{4.21}$$

Proof. We prove the case for $i = 1$. The proofs for $i = 2$ and $i = 3$ are similar. Note that v_{d_1} satisfies

$$\begin{cases} -D_i v_{d_1}'' - \alpha_1 v_{d_1}' = \left[g_1\left(\exp\left[-k_0 x - \int_0^x e^{(\alpha_1/D_1)y} v_{d_1}(y)\,dy \right] \right) - d_1 \right] v_{d_1}, & 0 < x < 1, \\ v_{d_1}'(0) = v_{d_1}'(1) = 0. \end{cases}$$

For any $(v_1, v_2, v_3) \in T_1 \cup T_2 \cup T_3$, v_1 satisfies

$$\begin{cases} -D_1 v_1'' - \alpha_1 v_1' \leqslant \left[g_1\left(\exp\left[-k_0 x - \int_0^x e^{(\alpha_1/D_1)y} v_1(y)\,dy \right] \right) - d_1 \right] v_1, & 0 < x < 1, \\ v_1'(0) = v_1'(1) = 0. \end{cases}$$

Choose $\theta(x) > v_1(x)$ for all $x \in [0, 1]$. It is not difficulty to prove that the parabolic problem

$$\begin{cases} v_t = D_1 v'' + \alpha_1 v' + \left[g_1\left(\exp\left[-k_0 x - \int_0^x e^{(\alpha_1/D_1)y} v(y)\,dy \right] \right) - d_1 \right] v, \\ v_x(0, t) = v_x(1, t) = 0, \quad v(x, 0) = \theta(x) \end{cases} \tag{4.22}$$

has a unique positive solution $v(x, t)$ for all $t \geqslant 0$. We may now find a small $\delta > 0$ such that $v_1(x) < v(x, t)$ for all $x \in [0, 1]$ and $t \in [0, \delta)$. Note that $v_1(x, t) \equiv v_1(x)$ satisfies

$$\begin{cases} (v_1)_t \leqslant D_1 (v_1)_{xx} + \alpha_1 (v_1)_x = \left[g_1\left(\exp\left[-k_0 x - \int_0^x e^{(\alpha_1/D_1)y} v_1(y)\,dy \right] \right) - d_1 \right] v_1, \\ (v_1)_x(0, t) = (v_1)_x(1, t) = 0, \quad v_1(x, 0) = v_1(x). \end{cases} \tag{4.23}$$

By Lemma 4.1 of [9] we have

$$\int_0^x e^{(\alpha_1/D_1)y} v_1(y)\,dy < \int_0^x e^{(\alpha_1/D_1)y} v(y,t)\,dy \quad \text{for all } t \geqslant 0 \text{ and } x \in (0,1]. \tag{4.24}$$

By Theorem 2.2 of [9] we have

$$\lim_{t\to\infty} v(x,t) = v_{d_1}(x) \quad \text{uniformly for } x \in [0,1].$$

Then (4.21) follows by letting $t \to \infty$ in (4.24). □

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. From (3.13), we have

$$\begin{aligned} &0 < d_i < -\lambda_1^{(i)}\bigl[-g_i\bigl(\sigma_{d_1}(x)\bigr)\bigr], \quad i = 2,3,\\ &0 < d_j < -\lambda_1^{(j)}\bigl[-g_j\bigl(\sigma_{d_2}(x)\bigr)\bigr], \quad j = 3,1,\\ &0 < d_k < -\lambda_1^{(k)}\bigl[-g_k\bigl(\sigma_{d_3}(x)\bigr)\bigr], \quad k = 1,2. \end{aligned}$$

By Lemma 4.5 we have w_i, $i = 1,2,3$ are all isolated. Therefore $\mathrm{index}_C(A, w_i)$, $i = 1,2,3$ are all well-defined and

$$\mathrm{index}_C(A, w_i) = 0, \quad i = 1,2,3.$$

Moreover, by Lemma 4.6 T_1, T_2 and T_3 are all compact sets hence $\mathrm{index}_C(A, T_i)$, $i = 1,2,3$ are all well-defined.

For any $(v_1, v_2, v_3) \in T_1 \cup T_2 \cup T_3$, we have by Lemma 4.9

$$0 \leqslant \int_0^x e^{(\alpha_i/D_i)y} v_i(y) \leqslant \int_0^x e^{(\alpha_i/D_i)y} v_{d_i}(y), \quad x \in [0,1],\ i = 1,2,3. \tag{4.25}$$

Hence

$$d_3 < -\lambda_1^{(3)}\bigl[-g_3\bigl(\sigma_{d_1 d_2}(x)\bigr)\bigr] \leqslant -\lambda_1^{(3)}\bigl[-g_3\bigl(\bar\sigma(x)\bigr)\bigr], \quad \text{for any } (\bar v_1, \bar v_2, 0) \in T_3.$$

Thus, (4.18) implies

$$\mathrm{index}_C(A, T_3) = 0.$$

In the same way, we have

$$\mathrm{index}_C(A, T_1) = \mathrm{index}_C(A, T_2) = 0.$$

Therefore,

$$\sum_{i=1}^3 \mathrm{index}_C(A, w_i) + \sum_{i=1}^3 \mathrm{index}_C(A, T_i) = 0 \neq 1.$$

It follows from Lemma 4.8 that (3.1) has at least one positive solution.

The proof of Theorem 3.3 is now complete. □

5. The n-species case

Our methods for the three-species system can be generalized to the general n-species ($n \geqslant 3$) case. We only list the important results and give necessary explanations. The proofs of these results simple extensions of those of the three species case.

Theorem 5.1. *If* (1.8) *has a positive steady state solution, then*

$$d_i \in \left(0, d_i^*\right) \quad \text{where } d_i^* = -\lambda_1^{(i)}\left[-g_i\left(e^{-k_0 x}\right)\right], \; i = 1, 2, \ldots, n.$$

Theorem 5.2.

(i) *If there is an $i \in \{1, 2, \ldots, n\}$ such that $d_i > \int_0^1 g_i(e^{-k_0 x})\,dx$, then there exists a constant $D > 0$, such that if $\min\{D_1, D_2, \ldots, D_n\} \geqslant D$ then* (1.8) *has only the trivial steady state solution.*
(ii) *If $d_i \in (0, \int_0^1 g_i(e^{-k_0 x})\,dx]$ for all $i = 1, 2, \ldots, n$, then there exists a positive constant D such that if $\min\{D_1, D_2, \ldots, D_n\} \geqslant D$,* (1.8) *has no positive steady state solution except possibly when the following exceptional situation occurs: there exists a constant $c \geqslant 0$ such that*

$$c_1 = c_2 = \cdots = c_n = c,$$

where c_i is uniquely determined by

$$d_i = \int_0^1 g_i\left(e^{-(k_0 + c_i)x}\right) dx, \quad i = 1, 2, \ldots, n. \tag{5.1}$$

Theorem 5.3. *Suppose that*

$$0 < d_i < -\lambda_1^{(i)}\left[-g_i\left(\kappa_i(x)\right)\right], \quad i = 1, 2, \ldots, n, \tag{5.2}$$

where

$$\kappa_i(x) = e^{-k_0 x} \exp\left(\sum_{j \neq i} \int_0^x e^{(\alpha_j / D_j) y} v_{d_j}(y)\,dy\right), \quad i = 1, 2, \ldots, n,$$

and v_{d_j} is the unique positive solution of the equation

$$-D_j v'' - \alpha_j v' = \left[g_j\left(e^{-k_0 x - \int_0^x e^{(\alpha_j / D_j) y} v(y)\,dy}\right) - d_j\right] v, \quad v'(0) = v'(1) = 0, \; j = 1, 2, \ldots, n.$$

Then (1.8) *has at least one positive steady state solution.*

Theorem 5.4. *Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n \in R^1$ and at least $n - 1$ of the α_i $(i = 1, 2, \ldots, n)$ are nonpositive. Then we can choose suitable $(D_1, D_2, \ldots, D_n)$ and $(d_1, d_2, \ldots, d_n)$ such that* (5.2) *holds, and hence there is at least one positive steady state solution to* (1.8).

Proof. We use induction to prove the theorem. Without loss of generality, we may assume $\alpha_j \leqslant 0$, $j = 2, 3, \ldots, n$.

By Theorem 3.4, Theorem 5.4 is valid for $n=3$. Suppose Theorem 5.4 is valid for $m(\geqslant 3)$, we show it is also valid for $m+1$. Theorem 5.4 is valid for m means there exist suitable $(D_1,\dots,D_m)$ and $(d_1,\dots,d_m)$ such that

$$0<d_i<-\lambda_{D_i}\left[-g_i\left(\exp\left[-k_0x-\sum_{j=1,\dots,m,\,j\neq i}\int_0^x v_{D_j,d_j}(s)e^{(\alpha_j/D_j)s}\,ds\right]\right)\right],\quad i=1,\dots,m. \tag{5.3}$$

Choose $\epsilon>0$ such that

$$0<d_i<-\lambda_{D_i}\left[-g_i\left(\exp\left[-k_0x-\sum_{j=1,\dots,m,\,j\neq i}\int_0^x v_{D_j,d_j}(s)e^{(\alpha_j/D_j)s}\,ds-\epsilon\right]\right)\right],\quad i=1,\dots,m.$$

By (3.17) and (3.18) we can choose $d_{m+1}>0$ such that $g_{m+1}(1)-d_{m+1}>0$ is small enough such that

$$\int_0^1 v_{D_{m+1},d_{m+1}}(x)e^{(\alpha_{m+1}/D_{m+1})x}\,dx\leqslant\epsilon\quad\text{for any } D_{m+1}>0.$$

Thus for any $D_{m+1}>0$ we have for all $i=1,\dots,m$,

$$0<d_i<-\lambda_{D_i}\left[-g_i\left(\exp\left[-k_0x-\sum_{j=1,\dots,m+1,\,j\neq i}\int_0^x v_{D_j,d_j}(s)e^{(\alpha_j/D_j)s}\,ds\right]\right)\right]. \tag{5.4}$$

By (3.14), with $D_1,\dots,D_m$, $d_1,\dots,d_m,d_{m+1}$ fixed, we have

$$\lim_{D_{m+1}\to 0}-\lambda_{D_{m+1}}\left[-g_{m+1}\left(\exp\left[-k_0x-\sum_{j=1,\dots,m}\int_0^x v_{D_j,d_j}(s)e^{(\alpha_j/D_j)s}\,ds\right]\right)\right]=g_{m+1}(1).$$

Note that $d_{m+1}\in(0,g_{m+1}(1))$. We can choose D_{m+1} sufficiently small such that

$$0<d_{m+1}<-\lambda_{D_{m+1}}\left[-g_{m+1}\left(\exp\left[-k_0x-\sum_{j=1,\dots,m}\int_0^x v_{D_j,d_j}(s)e^{(\alpha_j/D_j)s}\,ds\right]\right)\right].$$

Since (5.4) is not affected by the choice of D_{m+1}, we have for all $i=1,\dots,m+1$,

$$0<d_i<-\lambda_{D_i}\left[-g_i\left(\exp\left[-k_0x-\sum_{j=1,\dots,m+1,\,j\neq i}\int_0^x v_{D_j,d_j}(s)e^{(\alpha_j/D_j)s}\,ds\right]\right)\right]. \tag{5.5}$$

The proof is complete. □

Similarly we can extend Theorem 3.5 to obtain

Theorem 5.5. *Suppose $D_1,D_2,\dots,D_n$ are fixed. We can choose suitable $\alpha_1,\alpha_2,\dots,\alpha_n$ and $d_1,d_2,\dots,d_n$ such that (5.2) holds, and hence there exists at least one positive steady state solution to (1.8).*

Acknowledgments

The authors thank Professor Yihong Du for his patient guidance. This work is partially completed during X. Zhang's visit to the University of New England. She thanks UNE for the warm hospitality.

The authors thank the anonymous referees' valuable suggestions and comments, which helps improve the presentation of the paper.

References

[1] R.A. Armstrong, R. McGree, Competitive exclusion, Amer. Naturalist 115 (1980) 151–170.
[2] F. Belgacem, Elliptic Boundary Value Problems with Indefinite Weights: Variational Formulations of the Principal Eigenvalue and Applications, Pitman Res. Notes Math. Ser., vol. 368, Longman Sci., 1997.
[3] E.N. Dancer, On the indices of fixed points of mappings in cones and applications, J. Math. Anal. Appl. 91 (1983) 131–151.
[4] E.N. Dancer, Y. Du, Positive solutions for a three-species competition system with diffusion – I. General existence results, Nonlinear Anal. 24 (1995) 337–357.
[5] Y. Du, Positive periodic solutions of a competitor–competitor–mutualist model, Differential Integral Equations 9 (1996) 1043–1066.
[6] Y. Du, S.B. Hsu, Concentration phenomena in a nonlocal quasi-linear problem modelling phytoplankton I: Existence, SIAM J. Math. Anal. 40 (2008) 1419–1440.
[7] Y. Du, S.B. Hsu, Concentration phenomena in a nonlocal quasi-linear problem modelling phytoplankton II: Limiting profile, SIAM J. Math. Anal. 40 (2008) 1441–1470.
[8] Y. Du, S.B. Hsu, On a nonlocal reaction–diffusion problem arising from the modelling of phytoplankton growth, SIAM J. Math. Anal. 42 (2010) 1305–1333.
[9] Y. Du, L. Mei, On a nonlocal reaction–diffusion–advection equation modeling phytoplankton dynamics, Nonlinearity 24 (2011) 319–349.
[10] U. Ebert, M. Arrays, N. Temme, B. Sommeijer, J. Huisman, Critical condition for phytoplankton blooms, Bull. Math. Biol. 63 (2001) 1095–1124.
[11] L. Hsiao, P. De Mottoni, Persistence in reaction–diffusion systems: Interaction of two predators and one prey, Nonlinear Anal. 11 (1987) 877–891.
[12] S.B. Hsu, Limiting behavior for competing species, SIAM J. Appl. Math. 34 (1978) 760–763.
[13] S.B. Hsu, S. Hubbell, P. Waltman, A mathematical theory for single-nutrient competition in continuous cultures of microorganisms, SIAM J. Appl. Math. 32 (1977) 366–383.
[14] S.B. Hsu, Y. Lou, Single phytoplankton species growth with light and advection in a water column, SIAM J. Appl. Math. 70 (2010) 2942–2974.
[15] J. Huisman, N.N. Pham Thi, D.K. Karl, B. Sommeijer, Reduced mixing generates oscillations and chaos in oceanic deep chlorophyll, Nature 439 (2006) 322–325.
[16] J. Huisman, F.J. Wessing, Light-limited growth and competition for light in well-mixed aquatic environments: An elementary model, Ecology 75 (1994) 507–520.
[17] J. Huisman, F.J. Wessing, Competition for nutrients and light in a mixed water column: A theoretical analysis, Amer. Naturalist 146 (1995) 536–564.
[18] G.E. Hustchison, The paradox of the plankton, Amer. Naturalist 95 (1961) 137–145.
[19] H. Ishii, I. Takagi, Global stability of stationary solutions to a nonlinear diffusion equation in phytoplankton dynamics, J. Math. Biol. 16 (1982) 1–24.
[20] N. Lakos, Existence of steady-state solutions for a one-predator two-prey system, SIAM J. Math. Anal. 21 (1990) 647–659.
[21] A. Leung, A study of 3-species prey–predator reaction–diffusions by monotone schemes, J. Math. Anal. Appl. 100 (1984) 583–604.
[22] L. Mei, X. Zhang, On a nonlocal reaction–diffusion–advection system modeling phytoplankton growth with light and nutrients, Discrete Contin. Dyn. Syst. Ser. B 17 (2012) 221–243.
[23] N. Shigesada, A. Okubo, Analysis of the self-shading effect on algal vertical distribution in natural waters, J. Math. Biol. 12 (1981) 311–326.
[24] F.J. Weissing, J. Huisman, Growth and competition in a lighted gradient, J. Theoret. Biol. 168 (1994) 323–336.